Superattracting fixed points of quasiregular mappings

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Introduction

- We will look at the behaviour of the iterates of a function near a “superattracting” fixed point.
- Throughout the talk we assume without loss of generality that the fixed point is at the origin: $f(0) = 0$.
- Our aim is to generalise well-known results of complex dynamics to a higher-dimensional setting.

Complex dynamics

For holomorphic functions $f : \mathbb{C} \to \mathbb{C}$, the behaviour of the iterates near fixed points is well understood.

A fixpoint at $0 = f(0)$ is called *superattracting* if $f'(0) = 0$. In this case, there exists a conformal map ϕ in a nhd of 0 such that

$$\phi \circ f \circ \phi^{-1}(z) = z^d \quad \text{for some } d \geq 2.$$

Iterating this gives $\phi \circ f^k \circ \phi^{-1}(z) = z^{d^k}$. So for fixed small z,

$$c_0 d^k \leq \log \frac{1}{|f^k(z)|} \leq c_1 d^k, \qquad \forall k \in \mathbb{N}. \tag{$*$}$$

Thus if z, w are both near 0, then there exists $\alpha > 1$ such that

$$\frac{1}{\alpha} < \frac{\log |f^k(z)|}{\log |f^k(w)|} < \alpha, \qquad \forall k \in \mathbb{N}. \tag{$**$}$$

Quasiregular mappings

Quasiregular mappings of $\mathbb{R}^n$ generalise holomorphic functions on $\mathbb{C}$.

Definition

Let U be a domain in $\mathbb{R}^n$. A continuous function $f\colon U \to \mathbb{R}^n$ is called *quasiregular* (qr) if $f \in W^1_{n,\mathrm{loc}}(U)$ and there exists $K_O \geq 1$ such that

$$\|Df(x)\|^n \leq K_O J_f(x) \qquad \text{a.e. in } U.$$

The smallest such K_O is called the *outer dilatation* $K_O(f)$.

When f is qr, there also exists $K_I \geq 1$ such that

$$J_f(x) \leq K_I \inf_{|v|=1} |Df(x)v|^n \qquad \text{a.e. in } U,$$

and the smallest such K_I is called the *inner dilatation*, $K_I(f)$.

We say that f is K-*quasiregular* if $K \geq \max\{K_I(f), K_O(f)\}$.

Local index and Hölder continuity

To describe the 'valency' or 'multiplicity' of a qr map f at x we use:

Definition

The *local index* $i(x, f)$ is the minimum value of $\sup_{y \in \mathbb{R}^n} \operatorname{card}(f^{-1}(y) \cap V)$ as V runs through all neighbourhoods of x.

So f is injective near x if and only if $i(x, f) = 1$.

Quasiregular maps satisfy a local Hölder estimate:

Theorem (Martio, Srebro)

Let f be qr and non-constant near 0*. Then there exist $A, B, \rho > 0$ such that, for $x \in B(0, \rho)$,*

$$A|x|^{\nu} \leq |f(x) - f(0)| \leq B|x|^{\mu},$$

where $\nu = (K_O(f) i(0, f))^{\frac{1}{n-1}}$ *and* $\mu = \left(\frac{i(0,f)}{K_I(f)}\right)^{\frac{1}{n-1}}$.

A special case: Uniformly quasiregular maps

- If every iterate f^k is K-quasiregular with the same K, then f is called *uniformly quasiregular* (uqr).
- For uqr maps, many concepts of complex dynamics transfer nicely.
- In particular, Hinkkanen, Martin & Mayer classified local dynamics near a fixed point at 0. They showed:

If $i(0, f) = 1$ then f is bi-Lipschitz near 0. Classified attracting / repelling / neutral analogously to holomorphic case.

If $i(0, f) \geq 2$ then 0 called 'superattracting' and $f^k \to 0$ uniformly on a nhd of 0.

Difficulties with local dynamics

What kinds of local dynamics are possible near a fixed point $0 = f(0)$ of a general (non-uniformly) qr map?

Case $i(0, f) = 1$ Includes all local diffeomorphisms f, so appears a very general problem.

Case $i(0, f) \geq 2$ Unlike holomorphic and uqr cases, non-injectivity does not imply attracting.

E.g., if $K \in \mathbb{N}$, the winding map $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(re^{i\theta}) = re^{iK\theta}$ is K-qr with $f(0) = 0$ and $i(0, f) = K$. However, $|f(z)| = |z|$, so 0 is not attracting.

But we'll see that things change when $i(0, f) > K_I(f)$...

Strongly superattracting fixed points

Let $0 \in U$ and $f\colon U \to \mathbb{R}^n$ be a non-constant quasiregular map.

Definition

We call 0 a *strongly superattracting fixed point* (ssfp) if $f(0) = 0$ and

$$i(0, f) > K_I(f).$$

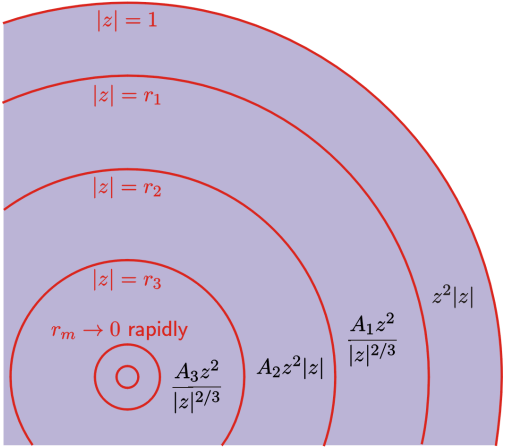
The basin of attraction in U is $\mathcal{A}(0) = \{x \in U : f^k(x) \in U,\ f^k(x) \to 0\}$.

If 0 is a ssfp, then $\mu = \left(\frac{i(0,f)}{K_I(f)}\right)^{\frac{1}{n-1}} > 1$, and so $f^k \to 0$ uniformly on a nhd of 0 by the Hölder estimate.

In fact, for small x, iterating the estimate gives $c_0, c_1 > 0$ such that

$$c_0 \mu^k \leq \log \frac{1}{|f^k(x)|} \leq c_1 \nu^k, \qquad \forall k \in \mathbb{N}.$$

An example



Define $g\colon \mathbb{D} \to \mathbb{C}$ as shown.

Then g is $\frac{3}{2}$-qr, in fact

$$K_I(g) = K_O(g) = \frac{3}{2}.$$

Also, $g(0) = 0$ with $i(0, g) = 2$.
So 0 is a ssfp.

$$\nu = 3 \text{ and } \mu = \frac{4}{3}.$$

We find that

$$\limsup_{k\to\infty} \left(\log \frac{1}{|g^k(z)|} \right)^{\frac{1}{k}} = 3 = \nu \quad \text{and} \quad \liminf_{k\to\infty} \left(\log \frac{1}{|g^k(z)|} \right)^{\frac{1}{k}} = \frac{4}{3} = \mu.$$

Main result

Notation: Denote a backward orbit by $O^-(x) := \bigcup_{k \geq 0} f^{-k}(x)$.

Theorem (Fletcher, N.)

Let $f\colon U \to \mathbb{R}^n$ be qr with a strongly superattracting fixed point at 0. If $x, y \in \mathcal{A}(0) \setminus O^-(0)$, then there exist $N \in \mathbb{N}$ and $\alpha > 1$ such that

(i)
$$|f^{k+N}(y)| < |f^k(x)|, \qquad \textit{for all large } k; \ \textit{and}$$

(ii)
$$\frac{1}{\alpha} < \frac{\log|f^k(x)|}{\log|f^k(y)|} < \alpha, \qquad \textit{for all large } k.$$

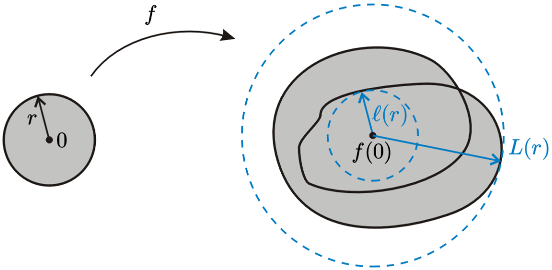
Can interpret (ii) as: "all orbits approach a ssfp at the same rate".

Sketch of proof

Define, for $r > 0$,

$$\ell(r) = \inf_{|x|=r} |f(x) - f(0)|,$$

$$L(r) = \sup_{|x|=r} |f(x) - f(0)|.$$



Proposition (FN refinement of GMRV result)

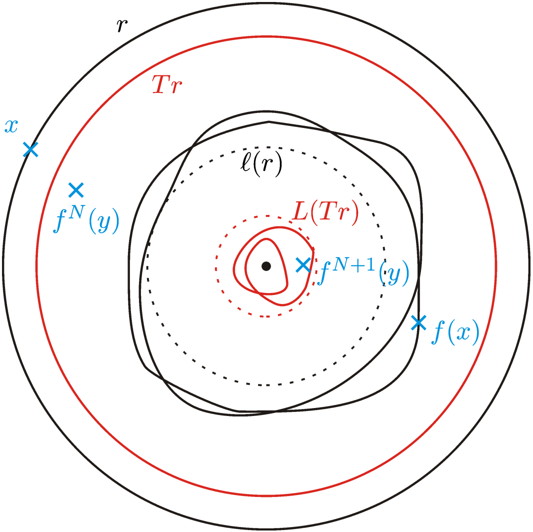
If f is quasiregular and non-constant on a nhd of 0, then there exists $C > 1$ such that for all $T \in (0,1)$ and small $r > 0$,

$$L(Tr) \le CT^{\mu} \ell(r), \qquad \textit{where } \mu = \left(\frac{i(0,f)}{K_I(f)} \right)^{\frac{1}{n-1}}.$$

When 0 is a ssfp, then $\mu > 1$ and we can fix T so small that $CT^{\mu} < T$. Prop then gives $L(Tr) \le T\ell(r)$ for all small r.

Given $x, y \in \mathcal{A}(0)$ near 0, find N such that $|f^N(y)| \leq T|x|$.
Then apply Prop with $r = |x|$ to get

$$|f^{N+1}(y)| \leq L(Tr) \leq T\ell(r) \leq T|f(x)|.$$



Iterating this idea gives

$$|f^{N+k}(y)| \leq T|f^k(x)|,$$

which proves (i).

Next, Hölder estimate for f^N gives $\alpha > 1$ such that

$$|f^k(y)|^\alpha < |f^N(f^k(y))| < |f^k(x)|,$$

which proves (ii):

$$\frac{\log|f^k(x)|}{\log|f^k(y)|} < \alpha. \quad \square$$

Polynomial type maps

Definition

A qr map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be of *polynomial type* if $\lim_{x\to\infty} |f(x)| = \infty$.

Fact: f is of polynomial type iff

$$\deg f := \max_{y\in\mathbb{R}^n} \operatorname{card} f^{-1}(y) < \infty.$$

- Can then extend to a qr map $f\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ by setting $f(\infty) = \infty$.
- We get that $i(\infty, f) = \deg f$.
- Hence ∞ is a strongly superattracting fixed point if $\deg f > K_I(f)$.
- Thus can restate our theorem in terms of the escaping set

$$I(f) = \{x \in \mathbb{R}^n : f^k(x) \to \infty \text{ as } k \to \infty\}.$$

Iteration of polynomial type maps

Theorem (Fletcher, N.)

Let f be a polynomial type qr map with $\deg f > K_I(f)$.
If $x, y \in I(f)$, then there are $N \in \mathbb{N}$ and $\alpha > 1$ such that, for all large k,

$$|f^k(x)| < |f^{k+N}(y)| \quad \textit{and} \quad \frac{1}{\alpha} < \frac{\log|f^k(x)|}{\log|f^k(y)|} < \alpha.$$

Fast escape

The *fast escaping set* $A(f) \subset I(f)$ can be defined as

$$A(f) = \{x \in \mathbb{R}^n : \exists N \in \mathbb{N}, |f^{k+N}(x)| > M^k(R, f) \text{ for all } k \in \mathbb{N}\},$$

where $M^k(R, f)$ denotes the iterated maximum modulus function.

- If f is trans entire on $\mathbb{C}$, then $\emptyset \neq A(f) \neq I(f)$. [Bergweiler-Hinkkanen, Rippon-Stallard]
- If f is trans type qr on $\mathbb{R}^n$, then $\emptyset \neq A(f) \neq I(f)$. [Bergweiler-Fletcher-Drasin / N.]
- If f is a complex polynomial on $\mathbb{C}$, then easy to see $A(f) = I(f)$.

What about polynomial type qr? Seems natural to restrict to $\deg f > K_I(f)$, else can have $I(f) = \emptyset$ (e.g. winding map).

Theorem (Fletcher, N.)

If f is qr of polynomial type with $\deg f > K_I(f)$, then $A(f) = I(f) \neq \emptyset$.