Connectedness properties of the set where the iterates of an entire function are unbounded

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1. Introduction

Denote the nth iterate of an entire function f by f^n, for n ∈ N. The Fatou set F(f) is the set of points z ∈ C such that the family of functions {f^n: n ∈ N} is normal in some neighbourhood of z, and the Julia set J(f) is the complement of F(f). For any z ∈ C, we call the sequence (f^n(z))_{n∈N} the orbit of z under f.

The set of points z ∈ C whose orbits are bounded is denoted by K(f). If f is a polynomial then K(f) is called the filled Julia set of f, though there is no widely used terminology if f is transcendental. Here, we are concerned with the complement of K(f), i.e. the set of points whose orbits are unbounded under iteration, K(f)ᶜ = {z ∈ C: (f^n(z))_{n∈N} is unbounded}. Clearly, K(f)ᶜ contains the escaping set I(f) of points whose orbits tend to infinity, I(f) = {z ∈ C: f^n(z) → ∞ as n → ∞}.

Indeed, if f is a polynomial it is well known that K(f)ᶜ = J(f), but if f is transcendental then there are always points in J(f) that are not in I(f) nor K(f), and there may also be points in the same property.

The properties of the escaping set for a general transcendental entire function were first investigated by Eremenko [1], and his conjecture that all components of K(f)ᶜ are transcendental. Here, we are concerned with the complement of K(f), i.e. the set of points whose orbits are unbounded under iteration, K(f)ᶜ = {z ∈ C: (f^n(z))_{n∈N} is unbounded}. Clearly, K(f)ᶜ contains the escaping set I(f) of points whose orbits tend to infinity, I(f) = {z ∈ C: f^n(z) → ∞ as n → ∞}.

Noting the simultaneous iteration of the minimum modulus of a transcendental entire function seems to be new. The next theorem gives a sufficient condition for a function to be transcendental, or else every neighbourhood of a point in J(f) meets uncountably many components of K(f)ᶜ.

Theorem 2 Let f be a transcendental entire function. Then:
(a) K(f)ᶜ is connected.
(b) Either K(f)ᶜ is connected, or else every neighbourhood of a point in J(f) meets uncountably many components of K(f)ᶜ.
(c) If K(f)ᶜ is connected, then K(f)ᶜ is connected.

We remark that Theorem 2(a) and (b) also hold for I(f). The result for I(f) corresponding to (a) was given in [6], and Rippon and Stallard have recently proved the result for I(f) corresponding to (b).

3. Iterating the minimum modulus

For a general transcendental entire function, the strongest partial result on Eremenko’s conjecture states that f(t) always has at least one unbounded component. This result was obtained by considering the minimal modulus of a transcendental entire function. The notation used here is explained.

Theorem 3 Let f be a transcendental entire function of order less than 1/2. Then there exists r > 0 such that m(r) → ∞ as n → ∞, and therefore K(f)ᶜ is connected.

5. Examples

The following examples illustrate some of our results.

A. If the subset A(f) of the fast escaping set A(f) takes the form of a sphere’s web, then Theorem 2 holds and K(f)ᶜ is connected. This also follows from Theorem 4(c) and results in [5], where the terminology used here is explained.

B. Let f(z) = 2 + e^{-z} and D₀ = {z: |Re z| ≤ (2^n+1)π, |Im z| ≤ (2^n+1)π}. Then A₀(f) is not a sphere’s web, but Theorem 2 holds and thus K(f)ᶜ is connected.

In fact, it can be shown that Theorem 1 also holds for this function.

C. Let f(z) = -100z e^{-z} - 4z. Then it can be shown that Theorem 2 holds but that Theorem 1 does not (see right hand box).

6. Example C

The function f maps the boundary of the domain D₀, shown in blue, onto the red curve, which lies entirely outside the domain D₀. Thus Theorem 2 holds, but note that m(r) → 0 as r → ∞, so the condition of Theorem 1 is not met.

4. Further connectedness properties of K(f)ᶜ

We note that K(f)ᶜ can be disconnected for a transcendental entire function f. For example, if f(z) = sin z then f maps the real line onto the interval [−1, 1], so it is a closed, connected set in K(f)ᶜ that disconnects K(f)ᶜ. For a general transcendental entire function we prove the following.

Theorem 4 Let f be a transcendental entire function. Then:
(a) K(f)ᶜ(∞) is connected.
(b) Either K(f)ᶜ is connected, or else every neighbourhood of a point in J(f) meets uncountably many components of K(f)ᶜ.
(c) If K(f)ᶜ is connected, then K(f)ᶜ is connected.

7. Key references