Connectedness properties of the set where the iterates of an entire function are unbounded

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1. Introduction

Denote the nth iterate of an entire function f by f^n, for $n \in \mathbb{N}$. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}$ such that the family of functions $\{f^n : n \in \mathbb{N}\}$ is normal in some neighbourhood of z, and the *Julia set* $J(f)$ is the complement of $F(f)$. For any $z \in \mathbb{C}$, we call the sequence $(f^n(z))_{n\in\mathbb{N}}$ the *orbit* of z under f.

The set of points $z \in \mathbb{C}$ whose orbits are bounded is denoted by $K(f)$. If f is a polynomial then $K(f)$ is called the *filled Julia set* of f, though there is no widely used terminology if f is transcendental. Here, we are concerned with the complement of $K(f)$, i.e. the set of points whose orbits are *unbounded* under iteration,

$$K(f)^c = \{z \in \mathbb{C} : (f^n(z))_{n\in\mathbb{N}} \text{ is unbounded}\}.$$

Clearly, $K(f)^c$ contains the *escaping set* $I(f)$ of points whose orbits tend to infinity,

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Indeed, if f is a polynomial it is well known that $K(f)^c = I(f)$, but if f is transcendental then there are always points in $J(f)$ that are in neither $I(f)$ nor $K(f)$, and there may also be points in $F(f)$ with the same property.

The properties of the escaping set for a general transcendental entire function were first investigated by Eremenko [1], and his conjecture that all components of $I(f)$ are unbounded has stimulated much subsequent research in transcendental dynamics. The conjecture remains open, though there have been several partial results - see for example [3, 4, 7]. In this light, it is of interest to ask what can be said about the connectedness properties of the larger set $K(f)^c$ for a transcendental entire function.

2. Functions for which $K(f)^c$ is connected

Let $m(r)$ denote the minimum modulus of the transcendental entire function f,

$$m(r) = m(r, f) := \min\{|f(z)| : |z| = r\},$$

and let $m^n(r)$ denote the nth iterate of the function $r \mapsto m(r)$. We prove the following result.

Theorem 1 *Let f be a transcendental entire function for which there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$. Then $K(f)^c$ is connected.*

More generally, we prove

Theorem 2 *Let f be a transcendental entire function, and $(D_n)_{n\in\mathbb{N}}$ be a sequence of bounded, simply connected domains such that*
(a) $f(\partial D_n)$ surrounds D_{n+1}, for $n \in \mathbb{N}$, and
(b) every disc centred at 0 is contained in D_n for sufficiently large n.
Then $K(f)^c$ is connected.

Many functions that meet the conditions of Theorem 2 are *strongly polynomial-like* in the sense defined in [2]. It is shown there that a transcendental entire function is strongly polynomial-like if and only if there exists a sequence of bounded, simply connected domains $(D_n)_{n\in\mathbb{N}}$ such that $f(\partial D_n)$ surrounds $\overline{D}_n$, for $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{C}$, and $\overline{D}_n \subset D_{n+1}$, for $n \in \mathbb{N}$. Note, however, that Theorem 2 does not hold for all strongly polynomial-like functions.

3. Iterating the minimum modulus

For a general transcendental entire function, the strongest partial result on Eremenko's conjecture states that $I(f)$ always has at least one unbounded component. This result was obtained by considering the *fast escaping set* $A(f)$, a subset of $I(f)$ defined in terms of the iterated maximum modulus function $M(r, f)$. By contrast, Theorem 1 shows how the connectedness properties of $K(f)^c$, a *superset* of $I(f)$, are related to a condition on the iterated *minimum* modulus function.

The notion of iterating the minimum modulus of a transcendental entire function seems to be new. The next theorem gives a sufficient condition for a function to meet the condition on $m(r)$ in Theorem 1. Recall that the *order* $\rho(f)$ of a transcendental entire function f is defined as $\rho(f) := \limsup_{r\to\infty} \dfrac{\log\log M(r, f)}{\log r}$.

Theorem 3 *Let f be a transcendental entire function of order less than $1/2$. Then there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$, and therefore $K(f)^c$ is connected.*

4. Further connectedness properties of $K(f)^c$

We note that $K(f)^c$ can be disconnected for a transcendental entire function f. For example, if $f(z) = \sin z$, then f maps the real line $\mathbb{R}$ onto the interval $[-1, 1]$, so $\mathbb{R}$ is a closed, connected set in $K(f)$ that disconnects $K(f)^c$. For a general transcendental entire function we prove the following.

Theorem 4 *Let f be a transcendental entire function. Then:*
(a) $K(f)^c \cup \{\infty\}$ is connected.
(b) Either $K(f)^c$ is connected, or else every neighbourhood of a point in $J(f)$ meets uncountably many components of $K(f)^c$.
(c) If $I(f)$ is connected, then $K(f)^c$ is connected.

We remark that Theorem 4(a) and (b) also hold for $I(f)$. The result for $I(f)$ corresponding to (a) was given in [6], and Rippon and Stallard have recently proved the result for $I(f)$ corresponding to (b).

5. Examples

The following examples illustrate some of our results.

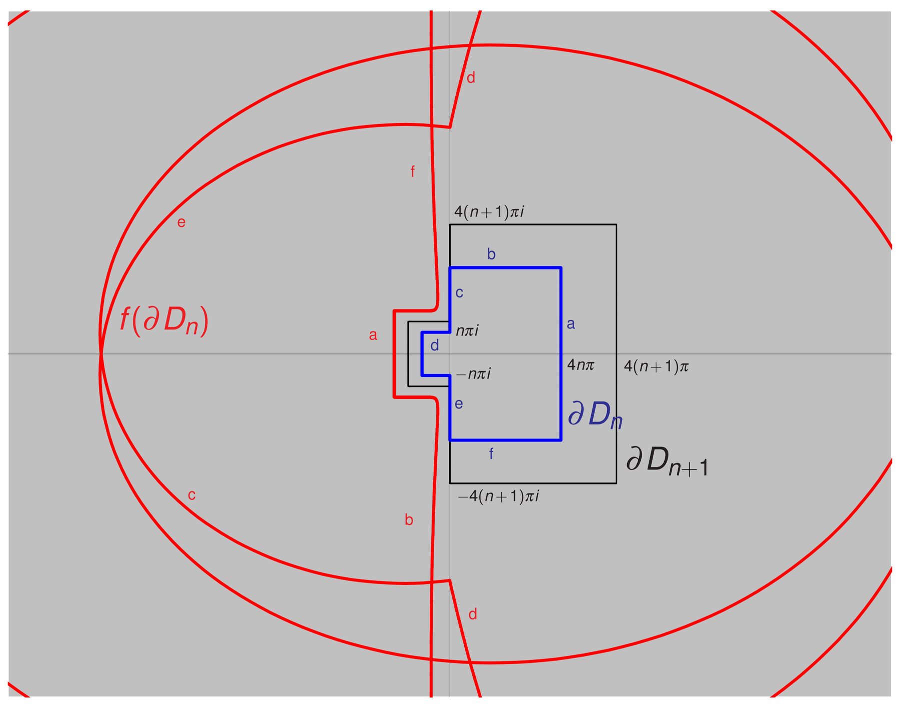
A. If the subset $A_R(f)$ of the fast escaping set $A(f)$ takes the form of a spider's web, then Theorem 2 holds and $K(f)^c$ is connected. This also follows from Theorem 4(c) and results in [5], where the terminology used here is explained.

B. Let $f(z) = 2z + e^{-z}$ and $D_n = \{z : |\text{Re } z| \le (2^n+1)\pi, |\text{Im } z| \le (2^n+1)\pi\}$. Then $A_R(f)$ is not a spider's web, but Theorem 2 holds and thus $K(f)^c$ is connected. In fact, it can be shown that Theorem 1 also holds for this function.

C. Let $f(z) = -10ze^{-z} - \frac{1}{2}z$. Then it can be shown that Theorem 2 holds but that Theorem 1 does not (see right hand box).

D. Let $f(z) = \cos z + z$. Then f is strongly polynomial-like, but the condition in Theorem 2(a) does not hold for any $(D_n)_{n\in\mathbb{N}}$. Indeed $\mathbb{R} \subset K(f)$, so it follows that $K(f)^c$ is disconnected.

6. Example C $\quad f(z) = -10ze^{-z} - \frac{1}{2}z$



The function f maps the boundary of the domain D_n, shown in blue, onto the red curve, which lies entirely outside the domain D_{n+1}. Thus Theorem 2 holds, but note that $m(r) \sim \frac{1}{2}r$ as $r \to \infty$, so the condition of Theorem 1 is not met.

7. Key references

[1] A.E. Eremenko, On the iteration of entire functions, *Dynamical systems and ergodic theory,* Banach Center Publications 23, Warsaw, 1989, 339–345.

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[5] P.J. Rippon and G.M. Stallard, Fast escaping points of entire functions. *Proc. London Math. Soc.*, 105 (4) (2012), 787–820.

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