Classification of Thurston maps with parabolic orbifolds

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Będlewo, 25 July 2014

Main results

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There exists an algorithm $\mathcal{A}$ which does the following. Let f and g be marked Thurston maps and assume that every element of the canonical geometrization of f has hyperbolic orbifold. The algorithm $\mathcal{A}$, given the combinatorial descriptions of f and g, outputs 1 if f and g are Thurston equivalent and 0 otherwise.

Thurston maps

Definition

A *Thurston map* is a pair (f, P_f) where $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving branched self-cover of $\mathbb{S}^2$ of degree $d_f \geq 2$ and P_f is a finite forward invariant set that contains all critical values of f.

In particular, the branched cover f must be postcritically finite.

Thurston equivalence

Definition

Two Thurston maps f and g are combinatorially equivalent if and only if there exist two homeomorphisms $h_1, h_2 \colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{S}^2, P_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_2} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f.

Thurston's theorem

Theorem (Thurston's Theorem)

A postcritically finite branched cover $f : \mathbb{S}^2 \to \mathbb{S}^2$ *(except* $(2, 2, 2, 2)$*-maps) is either Thurston-equivalent to a rational map* g *(which is then necessarily unique up to conjugation by a Möbius transformation), or* f *has a Thurston obstruction.*

Multicurves

- a closed curve γ is *essential* if every component of $\mathbb{S}^2 \setminus \gamma$ contains at least two points of P_f
- a *multicurve* is a finite collection of pairwise disjoint and non-homotopic essential simple closed curves

Thurston matrix and obstructions

Definition

Denote by $\mathcal{C}$ the set of all homotopy classes of essential simple closed curves. Define the Thurston linear operator $M\colon \mathbb{R}^{\mathcal{C}} \to \mathbb{R}^{\mathcal{C}}$ by setting

$$M(\gamma) = \sum_{f(\gamma_i)=\gamma} \frac{1}{\deg f|_{\gamma_i}} \gamma_i.$$

Every multicurve Γ has its associated *Thurston matrix* M_Γ which is the restriction of M to $\mathbb{R}^\Gamma$.

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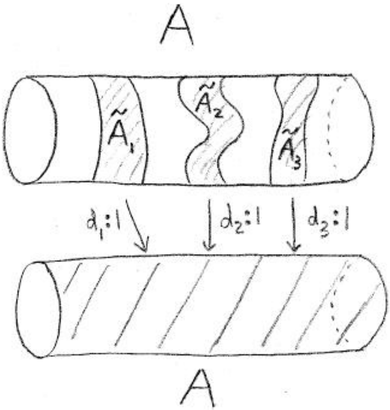
Definition

Since all entries of M_Γ are non-negative real, the leading eigenvalue λ_Γ of M_Γ is also real and non-negative. A multicurve Γ is a *Thurston obstruction* if $\lambda_\Gamma \geq 1$.

An example of Thurston obstruction



For a rational map, we must have $\sum 1/d_i < 1$.

Levy cycles

Definition

A *Levy cycle* is a multicurve

$$\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$$

such that each γ_i has a nontrivial preimage γ_i', where the topological degree of f restricted to γ_i' is 1 and γ_i' is homotopic to $\gamma_{(i-1) \mod n}$ rel Q. A Levy cycle is *degenerate* if each γ_i' bounds a disk D_i such that the restriction of f to D_i is a homeomorphism and $f(D_i)$ is homotopic to $D_{(i+1) \mod n}$ rel Q.

Algorithm for finding Thurston obstructions

Theorem (Bonnot, Braverman, Yampolsky)

There exists an algorithm which for any Thurston map f (which is not a (2,2,2,2)-map) outputs either an obstruction or an equivalent rational map.

Proof.

- Enumerate all possible multicurves and start checking if any of them is an obstruction for f one-by-one.
- List all (finitely many) rational maps that *could* be equivalent to f. List all homeomorphisms classes and check whether any of them realizes equivalence one-by-one.

□

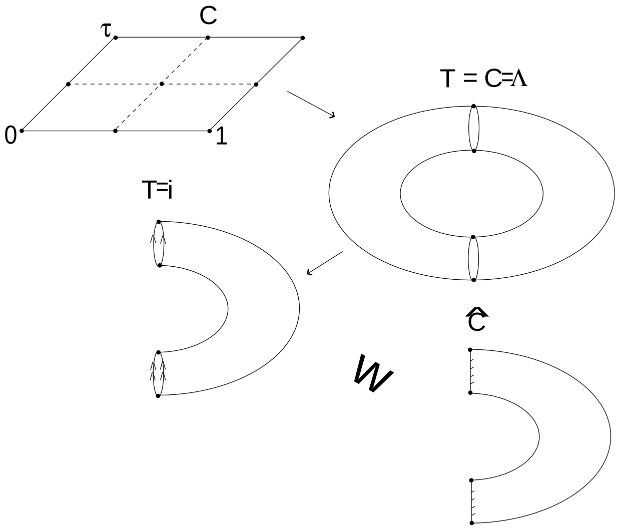
Decidability of combinatorial equivalence

Conjecture

There exists an algorithm which can produce a combinatorial equivalence between two Thurston maps or say that they are not equivalent.

Main Theorem 3 is a partial resolution of this conjecture.

$(2,2,2,2)$-maps



Classification of $(2, 2, 2, 2)$-maps

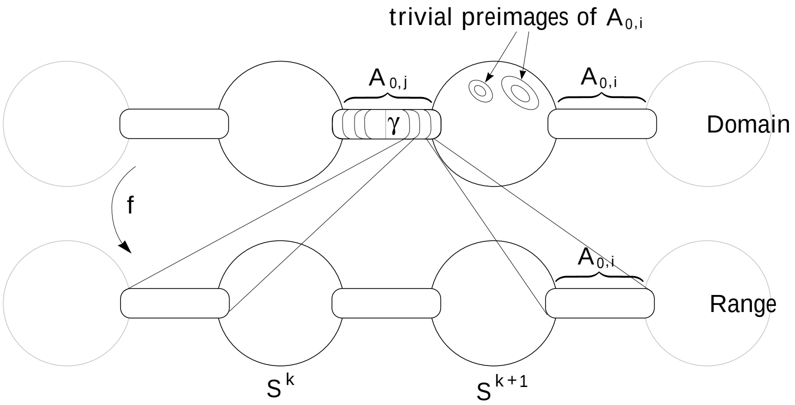
Theorem (Main Theorem 2)

Let f be a $(2, 2, 2, 2)$-map (with extra marked points) such that the associated matrix is hyperbolic. Then either f is equivalent to a quotient of an affine map or f admits a degenerate Levy cycle.
Furthermore, in the former case the affine map is defined uniquely up to conjugacy.

Corollary

There exists an algorithm which for any $(2, 2, 2, 2)$-map f with hyperbolic matrix outputs either a degenerate Levy cycle or an equivalent quotient of an affine map.

Pilgrim's decomposition of a Thurston map



Characterization of Canonical Thurston Obstructions

Theorem

The canonical obstruction Γ *is a unique minimal Thurston obstruction with the following properties.*

- *If the first-return map* F *of a cycle of components in* $\mathcal{S}_\Gamma$ *is a* $(2,2,2,2)$*-map, then every curve of every simple Thurston obstruction for* F *has two postcritical points of* f *in each complementary component and the two eigenvalues of* $\hat{F}_*$ *are equal or non-integer.*
- *If the first-return map* F *of a cycle of components in* $\mathcal{S}_\Gamma$ *is not a* $(2,2,2,2)$*-map or a homeomorphism, then there exists no Thurston obstruction of* F*.*

Computing Canonical Obstructions

Theorem

There exists an algorithm which for any Thurston map f finds its canonical obstruction Γ_f.

Proof.

1. Run the BBY algorithm to get an obstruction Γ.
2. Decompose f along Γ.
3. Check conditions of the previous theorem. Either they are satisfied or we can construct an obstruction within one of the decomposition pieces.
4. Once we have found a maximal obstruction we check the conditions of the characterization theorem for all of its subsets.

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Definition

Let f be a $(2,2,2,2)$-map and let z be an f-periodic point with period n. Fix a universal cover F of f and take a point $\tilde{z}$ in the fiber of z. If $z \notin P$, we define the *Nielsen index* $\mathrm{ind}_{F,n}(\tilde{z})$ to be the unique element g of the orbifold group G such that $F^n(\tilde{z}) = g \cdot \tilde{z}$. If $z \in P$ then the Nielsen index of z is defined up to pre-composition with the symmetry around z.

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Definition

Let f be a $(2,2,2,2)$-map and let z_1, z_2 be f-periodic points with period n. We say that z_1 and z_2 are in the same *Nielsen class of period n* if there exists a universal cover F_n of f^n and points $\tilde{z}_1, \tilde{z}_2$ in the fibers of z_1, z_2 respectively, such that both $\tilde{z}_1$ and $\tilde{z}_2$ are fixed by F_n.

Strategy of the proof

- A map f admits a degenerate Levy cycle if and only if there exist two distinct periodic points in P_f in the same Nielsen class.
- If there are points in the same Nielsen class, one can find a curve that separates them from other marked points which will generate a degenerate Levy cycle.
- If all points have distinct Nielsen indexes, they define a conjugacy between f and the appropriate quotient of an affine map on Q. It can be shown that in the absence of Levy cycles such a conjugacy can be promoted to a conjugacy on the whole sphere.