Lower bounds for real solutions to systems of polynomials

Perspectives of Modern Complex Analysis

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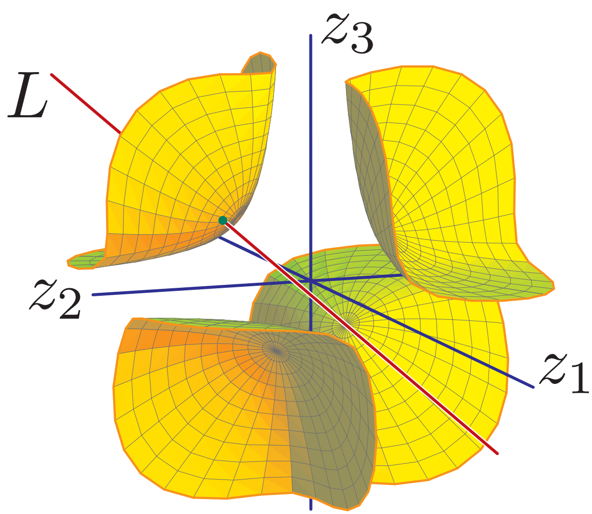
Frank Sottile

sottile@math.tamu.edu



Alex Eremenko

Frank Sottile, Texas A&M University

Inverse Wronski problem

The *Wronskian* of a (linear space of) univariate polynomials $f_1(t), \ldots, f_k(t)$ of degree $< n$ is the determinant

$$Wr(f_1(t), \ldots, f_k(t)) \;:=\; \det\left(\left(\frac{d}{dt}\right)^i f_j(t)\right)\,,$$

which has degree $k(n-k)$ (and is considered up to a scalar).

Inverse Wronski problem: Given a (real) polynomial $F(t)$ of degree $k(n-k)$, which linear spaces have Wronskian $F(t)$?

Schubert (1884) and Eisenbud and Harris (1984) determined the number of complex spaces,

$$[k(n-k)]!\frac{1!\cdot 2!\cdots(k-1)!}{(n-1)!(n-2)!\cdots(n-k)!}\,.$$

Shapiro Conjecture

Conjecture (B. Shapiro & M. Shapiro c. 1994)

If $F(t)$ has all $k(n-k)$ roots real, then all k-dimensional linear spaces of polynomials with Wronskian $F(t)$ are real.

This conjecture posits a large class of systems of polynomial equations with real coefficients that have only real solutions.

This was intensively studied, not only theoretically, but also experimentally on computers. Many special cases were proven.

Eremenko-Gabrielov Theorem

Theorem (A. Eremenko & A. Gabrielov, c. 2001)
$(k=2)$ *If* $F(t)$ *has all roots real, then all* $\mathbf{2}$*-dimensional linear spaces of polynomials with Wronskian* $F(t)$ *are real.*

Eremenko-Gabrielov Theorem

Theorem (A. Eremenko & A. Gabrielov, c. 2001)
$(k = 2)$ *If $F(t)$ has all roots real, then all 2-dimensional linear spaces of polynomials with Wronskian $F(t)$ are real.*

$Wr(f(t), g(t)) = f'(t)g(t) - g'(t)f(t) \;=\; 0$ are critical points of the rational function $\varphi(t) := f(t)/g(t)$.

Alex: I know *everything* about rational functions.....

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Theorem (A. Eremenko & A. Gabrielov)

A rational function whose critical points lie on a circle maps that circle to a circle.

The proof used complex analysis (uniformization theorem), and I think I understood it.

Mukhin-Tarasov-Varchenko Theorem

Theorem (Mukhin, Tarasov, Varchenko, c. 2006) *If $F(t)$ has all $k(n-k)$ roots real, then all k-dimensional linear spaces of polynomials with Wronskian $F(t)$ are real.*

The methods were diverse and deep, from differential equations to mathematical physics (Bethe Ansatz), representation theory, and quantum groups.

The *coup-de-grace* was a real symmetric matrix each of whose real eigenvalues gave a real space of polynomials. You will hear more later today from Tarasov.

I cannot say that I really understand this proof.

The Wronski map, again

Identifying $\mathbb{C}^m$ with polynomials of degree $< m$, get maps

$$
\begin{array}{rclcll}
Wr : & \mathrm{Gr}(k, \mathbb{C}^n) & \longrightarrow & \mathbb{P}^{k(n-k)} & \text{(finite map)} \\
Wr_{\mathbb{R}} : & \mathrm{Gr}(k, \mathbb{R}^n) & \longrightarrow & \mathbb{RP}^{k(n-k)} & \\
 & \mathbb{R}^{k(n-k)} & \longrightarrow & \mathbb{R}^{k(n-k)} & \text{(proper map)}
\end{array}
$$

MTV Theorem: The inverse image of a polynomial with only real roots lies in the real Grassmannian, $\mathrm{Gr}(k, \mathbb{R}^n)$.

Eremenko-Gabrielov (c. 2001): If $Wr_{\mathbb{R}}$ had a topological degree, that would be a lower bound on the number of solutions to the real inverse Wronski problem, which was an approach to the Shapiro Conjecture.

Lower bounds for Wronski problem

If n is odd and $2k < n$, set $\sigma_{k,n}$ to be

$$\frac{1!2!\cdots(k-1)!(n-k-1)!(n-k-2)!\cdots(n-2k+1)!(\frac{k(n-k)}{2})!}{(n-2k+2)!\cdots(n-4)!(n-2)!\left(\frac{n-2k+1}{2}\right)!\cdots\left(\frac{n-3}{2}\right)!\left(\frac{n-1}{2}\right)!}\,.$$

Set $\sigma_{k,n} = 0$ if n is even. If $2k > n$, then set $\sigma_{k,n} := \sigma_{n-k,n}$.

Eremenko-Gabrielov. *The topological degree of the proper map* $Wr : \mathbb{R}^{k(n-k)} \to \mathbb{R}^{k(n-k)}$ *is* $\sigma_{k,n}$.

Consequently, there are at least $\sigma_{k,n}$ real k-planes of polynomials of degree $< n$ with Wronskian a given general polynomial $F(t)$ of degree $k(n-k)$.

Why lower bounds are exciting

Many problems in engineering and science may be formulated as the solutions to a system of polynomial equations,

$$f_1(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,.$$

Typically, only the real or the positive solutions are meaningful for the application.

While the number of complex solutions is often known, *a priori* information on the real solutions is hard to obtain.

A non-trivial lower bound on the number of real solutions gives an existence proof for real solutions, which often suffices for the application.

Extending the scope of problems for which we have lower bounds will be important for the applications of mathematics.

Lower bounds from topology

Eremenko and Gabrielov used topology to get lower bounds on the number of real solutions to systems of polynomials.

Suppose that the real solutions are the fiber of a proper map

$$f^{-1}(x) \qquad \text{where} \qquad f\ :\ Y\ \longmapsto\ \mathbb{S}\,,$$

with Y and $\mathbb{S}$ oriented and $x \in \mathbb{S}$ is a regular value of f.

Then f has a well-defined *degree*, which is the weighted sum

$$\deg(f)\ :=\ \sum_{y\in f^{-1}(x)} \text{sign}\det df(y)\,.$$

(This sum is independent of the regular value x.)
Thus $|\deg(f)|$ is a lower bound on the number of solutions.

Sparse polynomials

A polynomial with support $\mathcal{A} \subset \mathbb{Z}^n$ is a sum

$$f \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha \qquad c_\alpha \in \mathbb{R}\,,$$

where $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
This is the pullback of a linear form $\sum c_\alpha z_\alpha$ along the map

$$\varphi \;:\; (\mathbb{C}^*)^n \ni x \longmapsto [x^\alpha \mid \alpha \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}}\,.$$

Set $X_\mathcal{A} := \overline{\varphi((\mathbb{C}^*)^n)}$ (a toric variety). A system of polynomials with support $\mathcal{A}$ corresponds to a linear section of $X_\mathcal{A}$,

$$f_1 \;=\; \cdots \;=\; f_n \;=\; 0 \quad\longleftrightarrow\quad X_\mathcal{A} \cap L\,,$$

and real solutions are real points in the section.

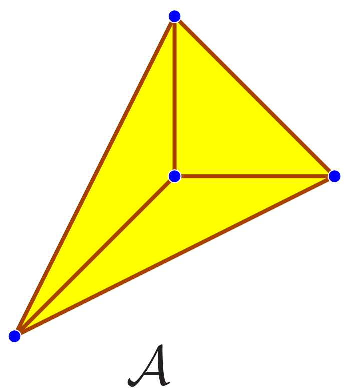
An example

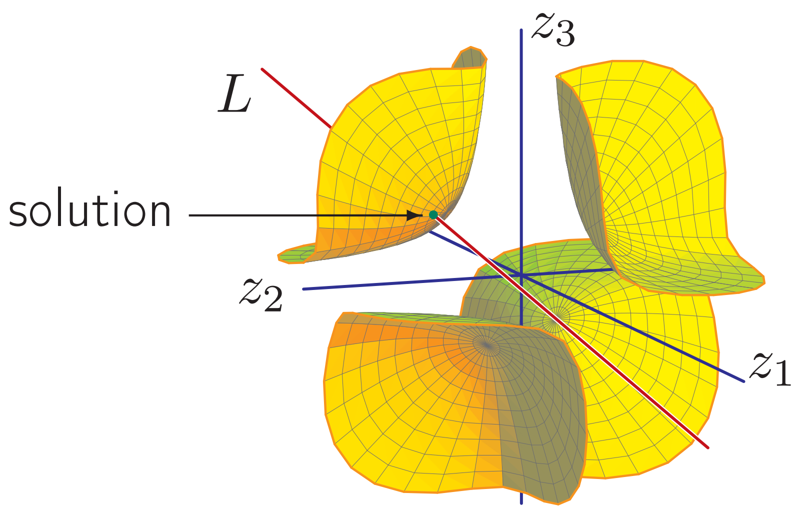
The system of polynomials

$$x^2y + 2xy^2 + xy - 1 \;=\; x^2y - xy^2 - xy + 2 \;=\; 0\,,$$

corresponds to a linear section of the toric variety

$$X_{\mathcal{A}} \;:=\; \overline{[xy : x^2y : xy^2 : 1]} \;=\; \mathcal{V}(z_1z_2z_3 - z_0^3)$$



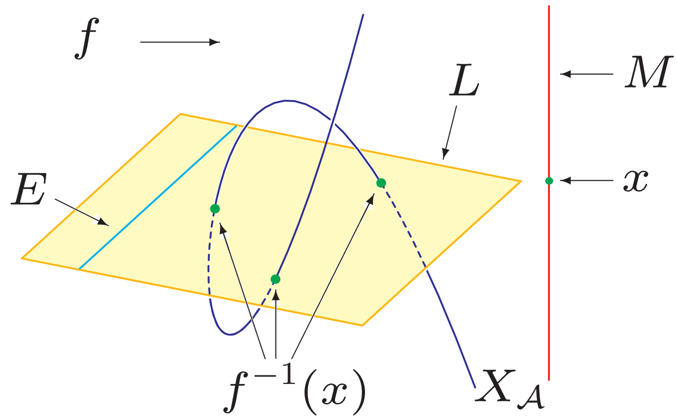


Polynomial systems as fibers

We realize $X_{\mathcal{A}} \cap L$ as the fiber of a map.

Let $E \subset L$ be a codimension one linear subspace and $M \simeq \mathbb{P}^n$ a complementary linear space.

The projection f from E sends $X_{\mathcal{A}}$ to M with $X_{\mathcal{A}} \cap L$ the fiber above $x = L \cap M$.



Restricting to $Y_{\mathcal{A}} := X_{\mathcal{A}} \cap \mathbb{RP}^{\mathcal{A}}$, the real solutions are fibers of

$$f \,:\, Y_{\mathcal{A}} \;\to\; M \cap \mathbb{RP}^{\mathcal{A}} \simeq \mathbb{RP}^n\,.$$

If $Y_{\mathcal{A}}$ and $\mathbb{RP}^n$ were orientable, $|\deg(f)|$ is a lower bound.

Orientability of real toric varieties

$Y_{\mathcal{A}}$ and $\mathbb{RP}^n$ are typically *not* orientable. This is improved by pulling back to the spheres $\mathbb{S}^{\mathcal{A}}$ and $\mathbb{S}^n$, which are oriented:

$$\begin{array}{ccccccc} f^+ : & Y_{\mathcal{A}}^+ & \subset & \mathbb{S}^{\mathcal{A}} & \xrightarrow{\ f^+\ } & \mathbb{S}^n \\ & \downarrow & & \downarrow & & \downarrow \\ f \ : & Y_{\mathcal{A}} & \subset & \mathbb{RP}^{\mathcal{A}} & \xrightarrow{\ f\ } & \mathbb{RP}^n \end{array}$$

The orientability of $Y_{\mathcal{A}}^+$ is characterized using the Newton polytope of $\mathcal{A}$. (Details omitted)

When $Y_{\mathcal{A}}^+$ is orientable, $|\deg(f^+)|$ is our lower bound.

Soprunova and I used geometric combinatorics and Gröbner bases to compute this degree in many cases, including recovering and extending the result of Eremenko-Gabrielov.

An interpolation problem

We all know that two points determine a line, and the Greeks knew that five points in the plane determine a conic.

Parameter counting shows that there will be finitely many, N_d, plane rational curves of degree d interpolating $3d-1$ general points. By 1873, $N_3 = 12$ and $N_4 = 620$ were known, which is where matters stood until about 1990, when Kontsevich gave an elegant recursion for the number N_d using ideas from Gromov-Witten theory/quantum cohomology.

What about real rational curves of degree d interpolating $3d-1$ real points in the plane?

Kharlamov showed there were 8, 10, or 12 real plane cubics $(d = 3)$ interpolating 8 general points.

Tropical lower bounds

About 2002, Welschinger proved that the weighted sum of real rational curves (weights are the parity ± 1 of the number of nodes) interpolating $3d-1$ real points was a constant, W_d, now called the Welschinger invariant.

Itenberg, Kharlamov, and Shustin used the tropical correspondence theorem of Mikhalkin to show that

$$W_d \geq \frac{d!}{3} \quad \text{and} \quad \lim_{d\to\infty} \frac{\log W_d}{\log N_d} = 1\,.$$

Thus W_d is a non-trivial lower bound for the number of real rational curves interpolating $3d-1$ points in $\mathbb{RP}^2$.

Lines on Calabi-Yau Hypersurfaces

There are finitely many lines on a hypersurface of degree $2n-1$ in $\mathbb{P}^{n+1}$: specifically, 27 lines on a cubic surface and 2875 lines on a quintic threefold.....

At least three of the lines on a real cubic surface are real. Segre classified these lines as elliptic or hyperbolic, and Okonek-Teleman observed that $h - e = 3$.

Separately, Okonek-Teleman and Kharlamov-Finashin generalized Segre's work, associating an intrinsic sign $\epsilon(\ell) \in \{\pm 1\}$ to a real line ℓ on a real hypersurface X of degree $2n-1$ in $\mathbb{P}^{n+1}$, and showed that

$$\sum_{\ell \subset X} \epsilon(\ell)\,,$$

is independent of the hypersurface X and equals $(2n-1)!!$.

Ramification of linear series

A space $V = \text{Span}\{f_1, \ldots, f_k\}$ of univariate polynomials is a linear series of dimension $k-1$ and degree $n-1$ on $\mathbb{P}^1$.

The ramification of V at a point $x \in \mathbb{P}^1$ is the increasing sequence $\alpha = 0 = \alpha_1 < \alpha_n < \cdots < \alpha_k$ for which there is a basis $g_1, \ldots, g_k$ of V with $\alpha_i = \text{ord}_x(g_i)$. The Wronskian of V vanishes to order $\sum_i \alpha_i - i + 1$ at x.

The inverse Wronski problem more generally asks for linear series with particular ramification at particular points of $\mathbb{P}^1$ (the ramification chosen so there are finitely many linear series).

Eremenko and Gabrielov, again

Ramification $\{(\alpha^1, x_1), \ldots, (\alpha^m, x_m)\}$ is real if

$$\{(\alpha^1, x_1), \ldots, (\alpha^m, x_m)\} \;=\; \{(\alpha^1, \overline{x_1}), \ldots, (\alpha^m, \overline{x_m})\}\,,$$

as multisets. Its type records the numbers of real and complex conjugate pairs among the (λ^i, x_i).

A natural generalization of the lower bounds of Eremenko-Gabrielov is to seek lower bounds for this problem of linear series with real ramification that depends upon type.

With Nick Hein, we investigated this on a supercomputer in a smallish experiment. (Investigated 344 million instances of 756 ramification problems, using 549 GHz-years of computing.) We observed that such lower bound were ubiquitous.

A taste of our data

Frequency table for $(0 < 6), (0 < 2)^7 = 6$, with $(k, n) = (2, 8)$

$r_{0<2}$	Number of Real Solutions				Total
	0	2	4	6	
7				100000	100000
5			77134	22866	100000
3		47138	47044	5818	100000
1	8964	67581	22105	1350	100000

We do have a proof of this lower bound of $r_{0<2} - 1$, but most of the other lower bounds we observed in the experiment we did not understand, but Tarasov does—see his talk.

Wronski map for $(k, n) = (3, 6)$

Observed numbers of real spaces versus c := number of complex conjugate pairs of roots of $F(t)$. Note that $\sigma_{3,6} = 0$.

c	0	2	4	6	8	10	12	14	16	18	20
1		1099		7975		42235		9081		6102	
2		24495		30089		25992		5054		3632	
3		39371		35022		15924		3150		1990	
4				76117		14481		3754		1375	

c	22	24	26	28	30	32	34	36	38	40	42
1	8827		1597		4207		1343		172		17362
2	4114		955		1586		832		63		3188
3	2183		494		622		367		35		842
4	2925		271		364		204		32		477

A congruence modulo four

The obvious congruence modulo four was established with Nick Hein and Igor Zelenko. The Grassmannian $\mathrm{Gr}(n, 2n)$ of n planes in $\mathbb{C}^{2n}$ has two commuting involutions: complex conjugation and a symplectic involution (corresponding to transpose of a matrix), which comes from the natural symplectic form on univariate polynomials.

For ramification problems that were symmetric, and where a numerical criterion holds which implies these involutions act independently, we were able to prove this observed congruence modulo four, for then the non-real solutions came in orbits of size four.

С Днём Рождения!