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Title : An introduction to Dunkl theory

(*from harmonic analysis on symmetric spaces and integrable systems related to quantum many body problems to Hecke algebras and special functions related to root systems*)

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Abstract. Dunkl theory is a far reaching generalization of Fourier analysis and special function theory related to root systems. During the sixties and seventies, it became gradually clear that radial Fourier analysis on rank one symmetric spaces was closely connected with certain classes of special functions in one variable :

- Bessel functions in connection with radial Fourier analysis on Euclidean spaces,
- Jacobi polynomials in connection with radial Fourier analysis on spheres,
- Jacobi functions (i.e. the Gauss hypergeometric function ${}_2\mathrm{F}_1$) in connection with radial Fourier analysis on hyperbolic spaces.

See [3] for a survey. During the eighties, several attempts were made, mainly by the Dutch school (Koornwinder, Heckman, Opdam), to extend these results in higher rank (i.e. in several variables), until the discovery of Dunkl operators in the rational case and Cherednik operators in the trigonometric case. Together with q–special functions introduced by Macdonald, this has led to a beautiful theory, developed by several authors ([1]), which encompasses in a unified way harmonic analysis on all Riemannian symmetric spaces and spherical functions thereon :

- generalized Bessel functions on flat symmetric spaces, and their asymmetric version, known as the Dunkl kernel,
- Heckman–Opdam hypergeometric functions on positively or negatively curved symmetric spaces, and their asymmetric version, due to Opdam,
- Macdonald polynomials on affine buildings.

Beside Fourier analysis and special functions, this theory has also deep and fruitful interactions with

- algebra (double affine Hecke algebras),
- mathematical physics (Calogero-Moser-Sutherland models, quantum many body problems),
- probability theory (Feller processes with jumps).

There are already several surveys about Dunkl theory available in the literature :

- [6] about rational Dunkl theory (state of the art in 2002),
- [5] about trigonometric Dunkl theory (state of the art in 1998),
- [4] and [1] about Dunkl theory and affine Hecke algebras,
- [2] about probabilistic aspects of Dunkl theory (state of the art in 2006).

In this series of lectures, we aim at giving an updated overview of Dunkl theory, mostly of its analytic and algebraic aspects.

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[1] Among others, let us mention Cherednik, de Jeu, Dunkl, Heckman, Macdonald, Opdam, Pasquale, Rösler, Schapira, Stokman, Trimèche, Yor, Ørsted, ...

References

[1] Ivan Cherednik : *Double affine Hecke algebras*, London Math. Soc. Lect. Note Ser. 319, Cambridge Univ. Press, Cambridge, 2005, 434 pages.

[2] *Harmonic and stochastic analysis of Dunkl processes*, Piotr Graczyk, Margit Rösler & Marc Yor (eds.), Travaux en cours 71, Hermann (2008).

[3] T.H. Koornwinder : *Jacobi functions and analysis on noncompact semisimple Lie groups*, in *Special functions (group theoretical aspects and applications)*, R.A. Askey,T.H. Koornwinder & W. Schempp (eds.), Reidel (1984), 1–84.

[4] Ian G. Macdonald : *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts Math. 157, Cambridge Univ. Press, Cambridge, 2003, 175 pages.

[5] Eric M. Opdam : *Lecture notes on Dunkl operators for real and complex reflection groups*, Math. Soc. Japan Mem. 8, Math. Soc. Japan, Tokyo, 2000, 90 pages.

[6] Margit Rösler : *Dunkl operators (theory and applications)*, in *Orthogonal polynomials and special functions (Leuven, 2002)*, Erik Koelink & Walter Van Assche (eds.), Lect. Notes Math. 1817, Springer–Verlag, Berlin, 2003, 93-135.