WKB analysis and Stokes geometry of differential equations

Yoshitsugu Takei
RIMS, Kyoto University
takei@kurims.kyoto-u.ac.jp

Lectures in Summer School
September 7–12, 2015, Będlewo, Poland

Abstract

In physics, since the very beginnings of the quantum mechanics, the WKB approximation has been employed to obtain approximate eigenfunctions and solve the eigenvalue problems of Schrödinger equations. The (full-order) WKB approximations provide formal solutions (with respect to the Planck constant) of Schrödinger equations but, as they are divergent in almost all cases, they were not so often used in rigorous mathematical analysis. Around 1980, using the Borel resummed WKB solutions, A. Voros (*Ann. Inst. H. Poincaré*, **39**(1983), 211-338) studied successfully some spectral functions of quartic oscillators. After the pioneering work of Voros, F. Pham, E. Delabaere and others (cf., e.g., E. Delabaere and F. Pham: *Ann. Inst. H. Poincaré*, **71**(1999), 1-94) have developed this new kind of WKB analysis (sometimes called "exact WKB analysis" or "complex WKB analysis") based on the Borel resummation technique and, in particular, the theory of Ecalle's resurgent functions. At present it turns out that the exact WKB analysis is very efficient not only for eigenvalue problems of Schrödinger equations but also for the global study of differential equations in the complex domain.

In the lectures, mainly using some concrete and illuminating examples, we explain the basic theory of the exact WKB analysis, its application to the global study of differential equations in the complex domain, and some recent developments of the theory. In the former part of lectures, we consider the

exact WKB analysis for second order linear ordinary differential equations (cf., T. Kawai and Y. Takei: *Algebraic Analysis of Singular Perturbation Theory*, AMS, 2005): Starting from the definition of WKB solutions, we first introduce the Stokes geometry and Voros' connection formulas for Borel resummed WKB solutions, which play a crucially important role in the theory. Then we discuss its application to the computation of monodromy groups of Fuchsian equations and wall crossing formulas for WKB solutions with respect to the change of parameters contained in the equation. In the latter part of lectures, we consider generalizations of the exact WKB analysis in various directions such as WKB analysis for higher order linear ordinary differential equations, generalization to completely integrable systems, and so on. New Stokes curves introduced by Berk-Nevins-Roberts (*J. Math. Phys.*, **23**(1982), 988-1002) are central problems in discussing such generalizations.

Contents

- Exact WKB analysis for second-order linear ODEs
- WKB solutions, Stokes geometry, connection formulas
- Application to the computation of monodromy groups of Fuchsian equations
- Voros coefficients and wall-crossing formulas
- New Stokes curves and virtual turning points for higher-order ODEs
- Generalization to completely integrable systems