The hypergeometric function and WKB solutions

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Analytic, Algebraic and Geometric Aspects of Differential Equations
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Bedlewo

1. Introduction

- **The Gauss hypergeometric differential equation:**

$$x(1-x)\frac{d^2w}{dx^2} + (c-(a+b+1)x)\frac{dw}{dx} - abw = 0, \tag{1.1}$$

where $a, b, c \in \mathbb{C}$.

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where $a, b, c \in \mathbb{C}$**.**

- **The hypergeometric series (or function):** $(c \neq 0, -1, -2, \ldots)$

(1.2) $$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_n = a(a+1)\cdots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$**, etc.**

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⋄ **The radius of convergence** $= 1$.
Thus $F(a,b,c;x)$ **defines a holomorphic function on** $\{x; |x| < 1\}$.
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- ⋄ $F(a,b,c;x)$ **defines a holomorphic function on the universal covering of** $\mathbb{C} - \{0,1\}$.

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$$\begin{array}{ccccc} 0 & \text{and} & 1-c & \text{at} & x=0 \\ 0 & \text{and} & c-a-b & \text{at} & x=1 \\ a & \text{and} & b & \text{at} & x=\infty \end{array}$$

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- **Other solutions of** (1.1)**:**

$$z^{1-c}F(a+1-c,b+1-c,1-c;x), \quad F(a,b,a+b+1-c;1-x),$$

$$(1-x)^{c-a-b}F(c-a,c-b,c+1-a-b;1-z), \quad (1-x)^{c-a-b}F\Big(a,c-b,c;\frac{x}{1-x}\Big), \quad \dots$$

(Kummer's 24 solutions)

Exact WKB analysis of the hypergeometric differential equation

- **Introduce a large parameter η in (1.1) by setting**

$$a = \frac{1}{2} + \alpha\eta,\ b = \frac{1}{2} + \beta\eta,\ c = 1 + \gamma\eta \quad (\alpha, \beta, \gamma \in \mathbb{C}).$$

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$$w = x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\psi.$$

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where $Q = Q_0 + \eta^{-2}Q_1$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2}, \quad Q_1 = -\frac{x^2 - x + 1}{4x^2(x-1)^2}.$$

- **WKB solutions:**

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int^{x} S_{\text{odd}}\, dx \right),$$

where $S = S_{\text{odd}} + S_{\text{even}}$ is a formal solution to Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q,$$

$S_{\text{odd}} = \sum_{j=0}^{\infty} \eta^{-2j+1} S_{2j-1}$ **with** $S_{-1} = \sqrt{Q_0}$ **and** $S_{\text{even}} = \sum_{j=0}^{\infty} \eta^{-2j} S_{2j}$.

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QUESTION:

What is the relation between $F(a, b, c; x)$ and $\phi_\pm$?

2. Some basics of exact WKB analysis of HGDE

- **Our equation:**

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where $Q = Q_0 + \eta^{-2}Q_1$ **with**

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- **Assumption on parameters:** We assume $(\alpha,\beta,\gamma)\notin E_0\cup E_1\cup E_2$, where

$$E_0=\{(\alpha,\beta,\gamma)\in\mathbb{C}^3 \mid \alpha\,\beta\,\gamma\,(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma)=0\},$$
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⋄ **There are two simple zeros** a_0, a_1 **of** Q_0**: simple turning points of** (2.1).

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- **Borel summability**

  $\psi_\pm$ and $\psi_\pm^{(0)}$ are Borel summable on each Stokes region (Koike-Schäfke).

⋄ What is Borel summability?

For a formal series with an exponential term ($\alpha \in \mathbb{R} - \{0, -1, -2, \dots\}$):

$$\varphi(x, \eta) = \exp(\eta a(x)) \sum_{n=0}^{\infty} \varphi_n(x)\eta^{-\alpha-n}$$

($a(x)$, $\varphi_j(x)$: **holomorphic in an open set** $U \subset \mathbb{C}$),

its Borel transform is defined by

$$\varphi_B(x, y) := \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{\Gamma(\alpha + n)}(y + a(x))^{\alpha+n-1}.$$

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• **WKB solutions are Borel transformable:**
For a WKB solution ψ, its Borel transform ψ_B converges locally.

φ **is Borel summable if**

- φ_B **converges in** $\{(x,y)\,|\,x \in U,\ |y + a(x)| < R\}$ **for some** $R > 0$,
- φ_B **can be analytically continued to the set**

$$\{(x,y)\,|\,x \in U,\ |\mathrm{Im}\,(y + a(x))| < R,\ \mathrm{Re}\,(y + a(x)) > 0\}$$

and the integral

$$\Phi(x,\eta) := \int_{-a(x)}^{\infty} \varphi_B(x,y)\exp(-\eta y)dy$$

makes sense and defines an analytic function of $x \in U$ **for** $\eta >> 1$.

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The Borel sum $\Phi(x,\eta)$ has an asymptotic expansion

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Thus the Borel sum of a WKB solution automatically has an asymptotic expansion

• Dominance relation of WKB solutions

The WKB solution ψ_+ (or $\psi_+^{(0)}$) is dominant (resp. recessive) on a Stokes curve emanating from a turning point a if

$$\mathrm{Re}\int_a^x \sqrt{Q_0}\,dx \geq 0 \quad \left(\text{resp.}\quad \mathrm{Re}\int_a^x \sqrt{Q_0}\,dx \leq 0\right)$$

on the Stokes curve.

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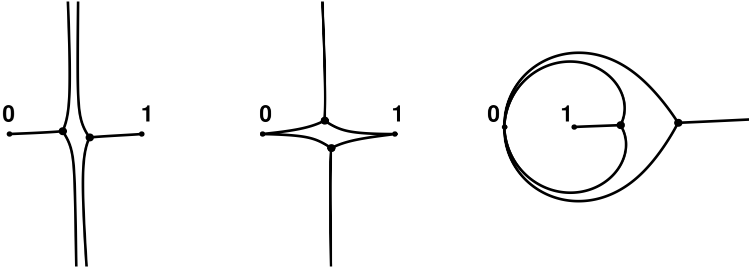
If ψ_+ (or $\psi_+^{(0)}$) is dominant (resp. recessive), then ψ_- (or $\psi_-^{(0)}$) is recessive (resp. dominant) on the Stokes curve.

- **Examples of Stokes curves of HGDE:**



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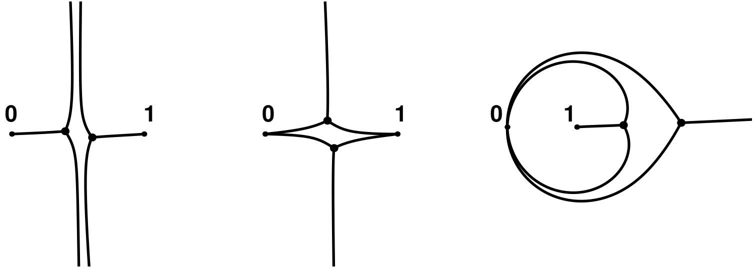

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Take the segments connecting two turning points as a branch cut and choose the branch of $\sqrt{Q_0}$ as $\sqrt{Q_0} \sim \dfrac{\gamma}{2x}$ $(\gamma = 1)$ near the origin.

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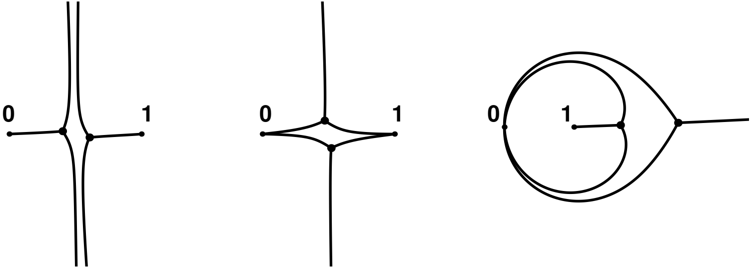

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Then ψ_+ (or $\psi_+^{(0)}$) is recessive on the Stokes curves which flow into the origin.

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Let n_0, n_1 and n_2 be the numbers of Stokes curves that flow into 0, 1 and ∞, respectively and let $\hat{n}$ denote (n_0, n_1, n_2). We define

$$
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\omega_3 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta\},\\
\omega_4 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha + \mathrm{Re}\,\beta < \mathrm{Re}\,\beta\},\\
G &= \text{the group generated by } \iota_m \quad (m = 0, 1, 2).
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Here ι_m $(m = 0, 1, 2)$ denote respectively the following involutions:

$$\iota_0 : (\alpha,\beta,\gamma) \mapsto (\beta,\alpha,\gamma),$$
$$\iota_1 : (\alpha,\beta,\gamma) \mapsto (\gamma-\alpha,\gamma-\beta,\gamma),$$
$$\iota_2 : (\alpha,\beta,\gamma) \mapsto (-\alpha,-\beta,-\gamma).$$

Set

$$\Pi_k = \bigcup_{r \in G} r(\omega_k) \quad (k = 1, \ldots, 4).$$

Theorem 1 (A-Tanda).

(1) If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$.
(2) If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.
(3) If $(\alpha, \beta, \gamma) \in \Pi_3$, then $\hat{n} = (1, 4, 1)$.
(4) If $(\alpha, \beta, \gamma) \in \Pi_4$, then $\hat{n} = (1, 1, 4)$.

Theorem 1 (A-Tanda).

(1) If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$.
(2) If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.
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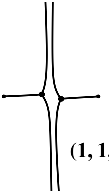
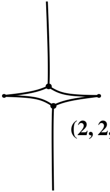
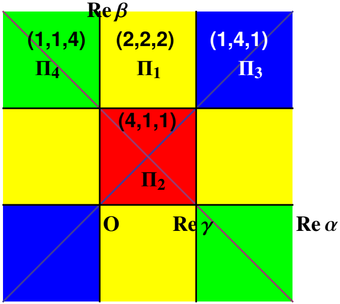


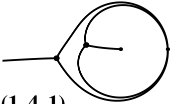
$(1, 1, 4)$



$(2, 2, 2)$



Reα-Reβ plane for a fixed Re $\gamma > 0$.



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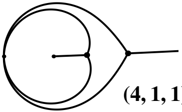

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$$V_0 := \int_0^{a_0} (S_{\rm odd} - \eta S_{-1}) dx$$ **the Voros coefficient of the origin**

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Similarly,

$$\psi_{\pm}^{(1)} = \exp(\pm V_1)\psi_{\pm} \qquad \psi_{\pm}^{(\infty)} = \exp(\pm V_{\infty})\psi_{\pm}$$

with

$$V_1 := \int_1^{a_0} (S_{\text{odd}} - \eta S_{-1})dx \quad \text{the Voros coefficient of 1,}$$

$$V_{\infty} := \int_{\infty}^{a_0} (S_{\text{odd}} - \eta S_{-1})dx \quad \text{the Voros coefficient of } \infty,$$

$$\psi_{\pm}^{(1)} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\Big(\pm \eta \int_{a_0}^{x} S_{-1}\, dx \pm \int_1^x (S_{\text{odd}} - \eta S_{-1})dx \Big),$$

$$\psi_{\pm}^{(\infty)} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\Big(\pm \eta \int_{a_0}^{x} S_{-1}\, dx \pm \int_{\infty}^x (S_{\text{odd}} - \eta S_{-1})dx \Big).$$

⋄ Explicit forms of the Voros coefficients

Theorem 2 (A-Tanda).

The Voros coefficients V_j ($j = 0, 1, \infty$) have the following forms:

$$V_0 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n(n-1)}\eta^{1-n}\left\{(1-2^{1-n})\left(\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}+\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right\},$$

$$V_1 = \pm\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n(n-1)}\eta^{1-n}\left\{(1-2^{1-n})\left(-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}+\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right\},$$

$$V_\infty = \pm\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n(n-1)}\eta^{1-n}\left\{(1-2^{1-n})\left(-\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}+\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}\right)-\frac{2}{(\beta-\alpha)^{n-1}}\right\}.$$

Here B_n ($n = 0, 1, 2, \ldots$) denote the Bernoulli numbers.

Lemma

The Voros coefficient V_0 is the unique solution the following system of difference equations as a formal power series solution in η^{-1} which is homogeneous in $(\alpha, \beta, \gamma, \eta^{-1})$ without constant term:

$$
\begin{aligned}
&V_0(\alpha+\eta^{-1},\beta,\gamma;\eta) - V_0(\alpha,\beta,\gamma;\eta) = \frac{1}{2}\log\frac{\gamma-\alpha-\frac{\eta^{-1}}{2}}{\alpha+\frac{\eta^{-1}}{2}} - \frac{\eta}{2}\Big\{\alpha\log\alpha \\
&\quad - (\alpha+\eta^{-1})\log(\alpha+\eta^{-1}) + (\gamma-\alpha)\log(\gamma-\alpha) - (\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})\Big\},\\
&V_0(\alpha,\beta+\eta^{-1},\gamma;\eta) - V_0(\alpha,\beta,\gamma;\eta) = \frac{1}{2}\log\frac{\gamma-\beta-\frac{\eta^{-1}}{2}}{\beta+\frac{\eta^{-1}}{2}} - \frac{\eta}{2}\Big\{\beta\log\beta \\
&\quad - (\beta+\eta^{-1})\log(\beta+\eta^{-1}) + (\gamma-\beta)\log(\gamma-\beta) - (\gamma-\beta-\eta^{-1})\log(\gamma-\beta-\eta^{-1})\Big\},\\
&V_0(\alpha,\beta,\gamma+\eta^{-1};\eta) - V_0(\alpha,\beta,\gamma;\eta) = \frac{1}{2}\log\frac{\gamma(\gamma+\eta^{-1})}{(\gamma-\alpha+\frac{\eta^{-1}}{2})(\gamma-\beta+\frac{\eta^{-1}}{2})}\\
&- \frac{\eta}{2}\Big\{(\gamma-\alpha)\log(\gamma-\alpha) - (\gamma-\alpha+\eta^{-1})\log(\gamma-\alpha+\eta^{-1}) + (\gamma-\beta)\log(\gamma-\beta)\\
&\qquad - (\gamma-\beta+\eta^{-1})\log(\gamma-\beta+\eta^{-1}) - 2\gamma\log\gamma + 2(\gamma+\eta^{-1})\log(\gamma+\eta^{-1})\Big\}.
\end{aligned}
$$

How to solve them?

We can rewrite them by using formal differential operators of infinite order:

$$V_0(\alpha + \eta^{-1}, \beta, \gamma; \eta) - V_0(\alpha, \beta, \gamma; \eta) = (e^{\eta^{-1}\partial_\alpha} - 1)V(\alpha, \beta, \gamma; \eta),$$
$$V_0(\alpha, \beta + \eta^{-1}, \gamma; \eta) - V_0(\alpha, \beta, \gamma; \eta) = (e^{\eta^{-1}\partial_\beta} - 1)V(\alpha, \beta, \gamma; \eta),$$
$$V_0(\alpha, \beta, \gamma + \eta^{-1}; \eta) - V_0(\alpha, \beta, \gamma; \eta) = (e^{\eta^{-1}\partial_\gamma} - 1)V(\alpha, \beta, \gamma; \eta).$$

The inverse operators:

$$(e^{\eta^{-1}\partial_\alpha} - 1)^{-1} = \eta\partial_\alpha^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \eta^{-n} \partial_\alpha^n, \text{ etc.,}$$

where the Bernoulli numbers B_n are defined by the generating function

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

⋄ Borel sums of V_j

Theorem 3 (A-Tanda).

V_j ($j = 0, 1, \infty$) is Borel summable in ω_k ($k = 1, 2, 3, 4$). The Borel sum V_j^k of V_j in ω_k has the following forms:

$$V_0^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}},$$

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\gamma\eta-1}},$$

$$V_0^3 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi},$$

$$V_0^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi},$$

$$V_1^1 = \pm\frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}}{\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma^2((\alpha+\beta-\gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta},$$

$$\cdots.$$

4. The relation between the hypergeometric function and WKB solutions

OUR QUESTION:

What is the relation between $F(a,b,c;x)$ and $\phi_{\pm}$?

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- **Basic idea**

To look at the recessive solutions on a Stokes curve: Recessive solutions form a one-dimensional subspace of the solution space.

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- **Basic idea**

To look at the recessive solutions on a Stokes curve: Recessive solutions form a one-dimensional subspace of the solution space.

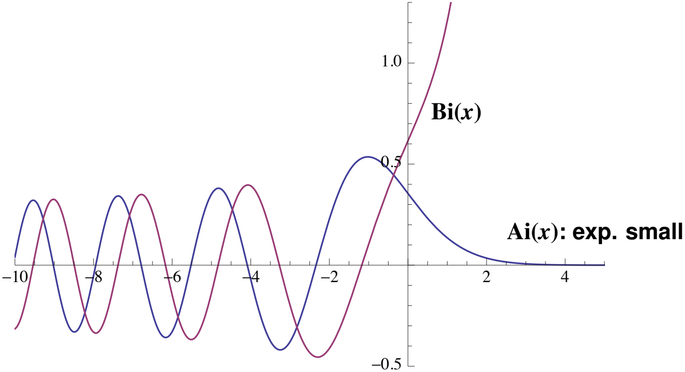
- **The simplest case: The Airy equation with a large parameter:**

$$\left(-\frac{d^2}{dx^2} + \eta^2 x\right)\psi = 0.$$

Standard analytic solutions: $\mathrm{Ai}(\eta^{\frac{2}{3}}x), \quad \mathrm{Bi}(\eta^{\frac{2}{3}}x).$

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp\left(\frac{1}{3}t^3 - xt\right)dt,$$

$$\mathrm{Bi}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty e^{\pi i/3}} \exp\left(\frac{1}{3}t^3 - xt\right)dt + \frac{1}{2\pi}\int_{-\infty}^{\infty e^{-\pi i/3}} \exp\left(\frac{1}{3}t^3 - xt\right)dt$$



Asymptotic expansion: $|\arg x| \le \pi - \delta, x \to \infty$

$$\mathrm{Ai}(\eta^{\frac{2}{3}}x) \sim \frac{1}{2\pi\eta^{\frac{1}{6}}x^{\frac{1}{4}}} \exp\left(-\frac{2}{3}\eta x^{\frac{3}{2}}\right) \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(3n+\frac{1}{2})}{(2n)!(3^2\eta x^{\frac{3}{2}})^n}$$

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WKB solutions:

$$\psi_{\pm}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_0^x S_{\mathrm{odd}} dx\right).$$

Here $S_{\mathrm{odd}} = \eta S_{-1} + \eta^{-1} S_1 + \eta^{-3} S_3 + \cdots$ with

$$S_{-1} = \sqrt{x}, \quad S_1 = -\frac{5}{32}x^{-\frac{5}{2}}, \quad S_3 = -\frac{1105}{2048}x^{-\frac{11}{2}}, \ldots.$$

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Hence

$$\frac{1}{\sqrt{S_{\mathrm{odd}}}} = \eta^{-\frac{1}{2}}x^{-\frac{1}{4}}\left(1 + O(\eta^{-2}x^{-\frac{7}{2}})\right),$$

$$\int_0^x S_{\mathrm{odd}}dx = -\frac{2}{3}\eta x^{\frac{3}{2}} + \eta^{-1}\frac{5}{48}x^{-\frac{3}{2}} + \cdots.$$

If we take the branch $\sqrt{x} > 0$ for $x > 0$, then

$$\psi_{-}(x,\eta) = \eta^{-\frac{1}{2}} x^{-\frac{1}{4}} \exp\Big(-\frac{2}{3}\eta x^{\frac{3}{2}}\Big)\Big(1 + O(\eta^{-1} x^{-\frac{3}{2}})\Big) \tag{4.1}$$

is a recessive solution and it is Borel summable with respect to η if $-\frac{2}{3}\pi < \arg x < \frac{2}{3}\pi$.

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Let $\Psi_{-}(x,\eta)$ denote the Borel sum of $\psi_{-}(x,\eta)$. Then $\Psi_{-}(x,\eta)$ is an exponentially small solution to the Airy equation. Hence there is a constant $C \neq 0$ such that

$$\mathrm{Ai}(\eta^{\frac{2}{3}} x) = C\,\Psi_{-}(x,\eta).$$

If we take the branch $\sqrt{x} > 0$ for $x > 0$, then

$$\psi_-(x,\eta) = \eta^{-\frac{1}{2}} x^{-\frac{1}{4}} \exp\Big(-\frac{2}{3}\eta x^{\frac{3}{2}}\Big)\Big(1 + O(\eta^{-1}x^{-\frac{3}{2}})\Big) \tag{4.1}$$

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Comparing (4.1) with the asymptotic expansion of the Airy function

$$\mathrm{Ai}(\eta^{\frac{2}{3}}x) \sim \frac{1}{2\pi\eta^{\frac{1}{6}}x^{\frac{1}{4}}} \exp\Big(-\frac{2}{3}\eta x^{\frac{3}{2}}\Big)\Gamma\Big(\frac{1}{2}\Big) \quad \text{as } x \to \infty,$$

we have $C = \dfrac{1}{2\sqrt{\pi}}\eta^{\frac{1}{3}}$, namely, $\quad \mathrm{Ai}(\eta^{\frac{2}{3}}x) = \dfrac{1}{2\sqrt{\pi}}\eta^{\frac{1}{3}}\,\Psi_-(x,\eta).$

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Let us recall the notation:

$$\phi_\pm := x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{\eta(\alpha+\beta-\gamma)}{2}}\Psi_\pm,$$

$$\Psi_\pm = \text{the Borel sum of } \psi_\pm.$$

Here

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{a_0}^{x} S_{\mathrm{odd}}\,dx\right)$$

are the WKB solutions of

(4.2)
$$\left(-\frac{d^2}{dx^2}+\eta^2 Q\right)\psi = 0,$$

where $Q = Q_0 + \eta^{-2}Q_1$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2}, \quad Q_1 = -\frac{x^2-x+1}{4x^2(x-1)^2}.$$

- **We consider the case where $(\alpha, \beta, \gamma) \in \omega_3$: $0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$**

Stokes geometry: type $(1, 4, 1)$

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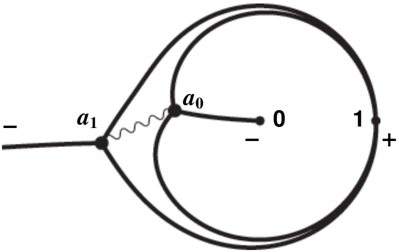

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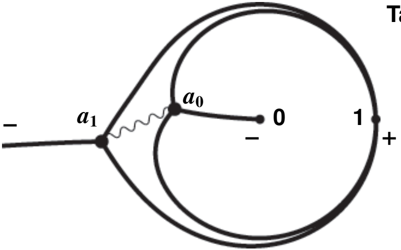

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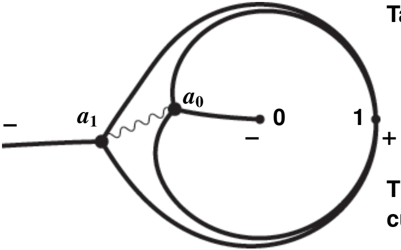

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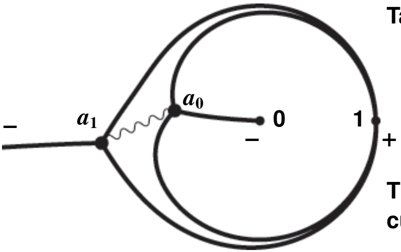

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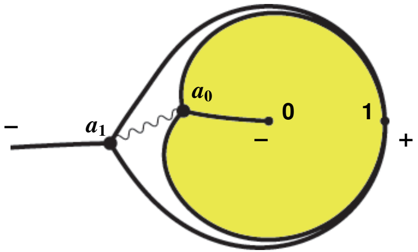
Then $\psi_-(x, \eta)$ is dominant on the Stokes curve flowing into the origin.

Thus the recessive solution $\psi_+(x, \eta)$ does not have Stokes phenomena on these Stokes curves.

This implies that . . .

$\psi_+(x,\eta)$ is Borel summable on the "cardioid-like" domain bounded by the Stokes curves emanating from a_0 and going to 1 (colored region) except for the origin (Koike-Schäfke).



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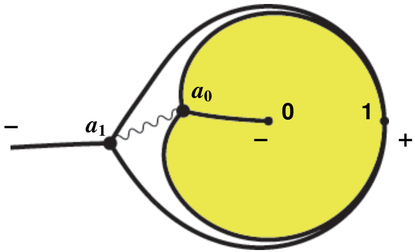

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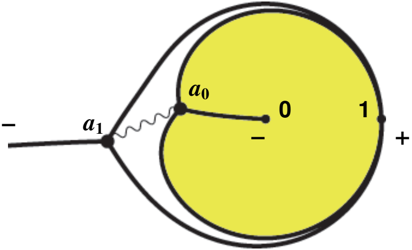

How about the origin?

The origin is a regular singular point of our equation.

But it plays a role of the infinity in the case of the Airy equation.

Let us examine the behavior of $\int_{a_0}^{x} \sqrt{Q_0}\, dx$ near the origin.

Recall that

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2} =: \frac{R(x)}{4x^2(x-1)^2}$$

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$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2} =: \frac{R(x)}{4x^2(x-1)^2}$$

By a little computation,

$$\int_{a_0}^{x} \sqrt{Q_0}\,dx = \frac{1}{4}\left((\gamma-\alpha-\beta)\log\frac{(p+2\delta^2)x + p + 2\gamma^2 + 2(\gamma-\alpha-\beta)\sqrt{R(x)}}{(p+2\delta^2)x + p + 2\gamma^2 - 2(\gamma-\alpha-\beta)\sqrt{R(x)}}\right.$$
$$\left. +\gamma\log\frac{px + 2\gamma^2 - 2\gamma\sqrt{R(x)}}{px + 2\gamma^2 + 2\gamma\sqrt{R(x)}} + \delta\log\frac{2\delta^2x + p - 2\delta\sqrt{R(x)}}{2\delta^2x + p + 2\delta\sqrt{R(x)}}\right).$$

Here $p = 2(2\alpha\beta - \beta\gamma - \gamma\alpha)$, $\delta = \beta - \alpha$.

Recall that

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2} =: \frac{R(x)}{4x^2(x-1)^2}$$

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$$\int_{a_0}^{x} \sqrt{Q_0}\,dx = \frac{1}{4}\left((\gamma-\alpha-\beta)\log\frac{(p+2\delta^2)x + p + 2\gamma^2 + 2(\gamma-\alpha-\beta)\sqrt{R(x)}}{(p+2\delta^2)x + p + 2\gamma^2 - 2(\gamma-\alpha-\beta)\sqrt{R(x)}}\right.$$
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Here $p = 2(2\alpha\beta - \beta\gamma - \gamma\alpha)$, $\delta = \beta - \alpha$.

By a little more computation . . .

$$\int_{a_0}^{x} \sqrt{Q_0}\, dx = \frac{\gamma}{2} \log x + h_0 + O(x)$$

with

$$h_0 = \frac{1}{2}(\alpha \log \alpha + \beta \log \beta + (\gamma - \alpha) \log(\alpha - \gamma) + (\gamma - \beta) \log(\beta - \gamma) - 2\gamma \log \gamma).$$

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- **The residue of $S_{\text{odd}}\, dx$ at the origin:**

$$\operatorname*{Res}_{x=0} S_{\text{odd}}\, dx = \operatorname*{Res}_{x=0} \eta S_{-1}\, dx = \frac{\eta \gamma}{2}$$

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- **The residue of $S_{\mathrm{odd}}\, dx$ at the origin:**

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}}\, dx = \operatorname*{Res}_{x=0} \eta S_{-1}\, dx = \frac{\eta\gamma}{2}$$

Hence we have

$$\psi_{+}^{(0)}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\eta \int_{a_0}^{x} S_{-1}\, dx + \int_{0}^{x} (S_{\mathrm{odd}} - \eta S_{-1}) dx\right)$$

$$= \sqrt{\frac{2}{\gamma}}\, \eta^{-\frac{1}{2}} x^{\frac{1}{2}+\frac{\gamma\eta}{2}} \exp(\eta h_0)\,(1 + O(x)).$$

We set $\tilde{\psi}_+^{(0)}(x,\eta) = x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}\psi_+^{(0)}(x,\eta)$.

Lemma 4.

(i) The formal series $\tilde{\psi}_+^{(0)}(x,\eta)$ is Borel summable in a neighborhood of the origin and the Borel sum $\tilde{\Psi}_+^{(0)}(x,\eta)$ of $\tilde{\psi}_+^{(0)}(x,\eta)$ is holomorphic there.

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Hence

$$\begin{aligned}\varphi_+^{(0)}(x,\eta) :=& (1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\tilde{\Psi}_+^{(0)}(x,\eta)\\ =& x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\Psi_+^{(0)}(x,\eta)\end{aligned}$$

is a holomorphic solution near the origin to the hypergeometric equation. Here $\Psi_+^{(0)}(x,\eta)$ is the Borel sum of $\psi_+^{(0)}(x,\eta)$.

There is a constant C_0 such that

$$F\Big(\frac{1}{2}+\alpha\eta, \frac{1}{2}+\beta\eta, 1+\gamma\eta; x\Big) = C_0\varphi_+^{(0)}(x,\eta).$$

Putting $x = 0$, we have

$$1 = C_0\sqrt{\frac{2}{\gamma}}\,\eta^{-\frac{1}{2}}\exp(\eta\, h_0),\ \text{ i.e., }\ C_0 = \sqrt{\frac{\gamma}{2}}\,\eta^{\frac{1}{2}}\exp(-\eta\, h_0).$$

Therefore

Theorem 5.

If $(\alpha,\beta,\gamma)\in\omega_3$, the following relation holds in a neighborhood of the origin:

$$F\Big(\frac{1}{2}+\alpha\eta, \frac{1}{2}+\beta\eta, 1+\gamma\eta; x\Big) = \sqrt{\frac{\gamma}{2}}\,\eta^{\frac{1}{2}}\exp(-\eta\, h_0)x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\Psi_+^{(0)}(x,\eta).$$

Here $h_0 = \dfrac{1}{2}(\alpha\log\alpha+\beta\log\beta+(\gamma-\alpha)\log(\alpha-\gamma)+(\gamma-\beta)\log(\beta-\gamma)-2\gamma\log\gamma).$

Remark $(\alpha,\beta,\gamma)\in\omega_3 \iff 0<\operatorname{Re}\gamma<\operatorname{Re}\alpha<\operatorname{Re}\beta$

The WKB solution $\psi_+(x,\eta)$ normalized at the turning point a_0 is related to $\psi_+^{(0)}(x,\eta)$ by using the Voros coefficient:

$$\psi_+^{(0)}(x,\eta) = \exp(V_0)\psi_+(x,\eta).$$

Taking the Borel sum of the both members for $(\alpha,\beta,\gamma) \in \omega_3$, we have

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Thus we have

Theorem 6.

If $\alpha,\beta,\gamma) \in \omega_3$, the hypergeometric function and the Borel sum of the WKB solution normalized at the turning point a_0 are related near the origin by

$$F\Big(\frac{1}{2}+\alpha\eta, \frac{1}{2}+\beta\eta, 1+\gamma\eta; x\Big) = \frac{\Big\{\Gamma\Big(\frac{1}{2}+(\alpha-\gamma)\eta\Big)\Gamma\Big(\frac{1}{2}+(\beta-\gamma)\eta\Big)\Big\}^{\frac{1}{2}}\Gamma(1+\gamma\eta)}{2\sqrt{\pi}\Big\{\Gamma\Big(\frac{1}{2}+\alpha\eta\Big)\Gamma\Big(\frac{1}{2}+\beta\eta\Big)\Big\}^{\frac{1}{2}}} x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{\eta(\alpha+\beta-\gamma)}{2}}\Psi_+(x,\eta).$$

A generalization

Our original setting:

$$x(1-x)\frac{d^2w}{dx^2} + (c-(a+b+1)x)\frac{dw}{dx} - abw = 0$$

with

$$a = \frac{1}{2} + \alpha\eta,\ b = \frac{1}{2} + \beta\eta,\ c = 1 + \gamma\eta \quad (\alpha, \beta, \gamma \in \mathbb{C}).$$

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What happens in the following case?

$$a = \alpha_0 + \alpha\eta,\ b = \beta_0 + \beta\eta,\ c = \gamma_0 + \gamma\eta \quad (\alpha, \alpha_0, \beta, \beta_0, \gamma, \gamma_0 \in \mathbb{C}).$$

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An easy going replacement

$$\alpha \to \alpha + (\alpha_0 - \frac{1}{2})\eta^{-1}, \beta \to \beta + (\beta_0 - \frac{1}{2})\eta^{-1}, \gamma \to \gamma + (\gamma_0 - 1)\eta^{-1}$$

in the results DOES NOT WORK straight!
We have to define and compute all the quantities, such as the Voros coefficients, h_0, etc., once again in this setting.

- **Eliminate the first-order term by setting**

$$w = x^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\psi.$$

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$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q\Big)\psi = 0,$$

where $Q = Q_0 + \eta^{-1}Q_1 + \eta^{-2}Q_2$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2},$$

$$Q_1 = \frac{(\alpha-\beta)(\alpha_0-\beta_0)x^2 + (2(\alpha\beta_0+\alpha_0\beta)-\beta\gamma_0-\beta_0\gamma-\gamma\alpha_0-\gamma_0\alpha+\gamma)x+\gamma(\gamma_0-1)}{2x^2(x-1)^2},$$

$$Q_2 = \frac{(\alpha_0-\beta_0+1)(\alpha_0-\beta_0-1)x^2 + 2(2\alpha_0\beta_0-\beta_0\gamma_0-\gamma_0\alpha_0+\gamma_0)x+\gamma_0(\gamma_0-1)}{4x^2(x-1)^2}.$$

The Riccati equation:

$$\frac{dS}{dx} + S^2 = \eta^2 Q$$

and its formal solution

$$S^{\pm} = \sum_{j=-1}^{\infty} \eta^{-j} S_j^{\pm} \quad \text{with} \quad S_{-1}^{\pm} = \pm \frac{\sqrt{(\alpha-\beta)^2 x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}}{2x(1-x)}.$$

- **WKB solutions:**

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int^x S_{\mathrm{odd}}\, dx \right),$$

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$$S_{\mathrm{odd}} := \frac{1}{2}(S^+ - S^-) = \sum_{j=-1}^{\infty} \eta^{-j} S_{\mathrm{odd},j}.$$

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Remark **We take the branch of $\sqrt{Q_0}$ as $\sqrt{Q_0} \sim \dfrac{\gamma}{2x}$ near the origin.**

$$S_{\mathrm{odd},-1} = S_{-1}^{+},$$

$$S_{\mathrm{odd},0} = \frac{(\alpha-\beta)(\alpha_0-\beta_0)x^2 + (2(\alpha\beta_0 + \alpha_0\beta) - \beta\gamma_0 - \beta_0\gamma - \gamma\alpha_0 - \gamma_0\alpha + \gamma)x + \gamma(\gamma_0 - 1)}{2x(1-x)\sqrt{(\alpha-\beta)^2 x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}}.$$

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⋄ **WKB solutions normalized at the regular singular point b ($b = 0, 1, \infty$):**

$$\psi_{\pm}^{(b)} = \frac{1}{\sqrt{S_{\rm odd}}} \exp\Big(\pm \int_{a_0}^{x} S_{{\rm odd},\le 0}\, dx \pm \int_{b}^{x} (S_{\rm odd} - S_{{\rm odd},\le 0}) dx \Big),$$

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where a_0 is a simple turning point and

$$S_{\mathrm{odd},\le 0} = \eta S_{\mathrm{odd},-1} + S_{\mathrm{odd},0}.$$

Remark: **$S_{\mathrm{odd}}\, dx$ and $S_{\mathrm{odd},\le 0}\, dx$ have a simple pole at $x = b$ and**

$$\operatorname*{Res}_{x=b} S_{\mathrm{odd}}\, dx = \operatorname*{Res}_{x=b} S_{\mathrm{odd},\le 0}\, dx.$$

The Voros coefficients:

$$V_0 := \int_{0}^{a_0} (S_{\mathrm{odd}} - S_{\mathrm{odd},\le 0}) dx,\ V_1 := \int_{1}^{a_0} (S_{\mathrm{odd}} - S_{\mathrm{odd},\le 0}) dx,$$

$$V_\infty := \int_{\infty}^{a_0} (S_{\mathrm{odd}} - S_{\mathrm{odd},\le 0}) dx.$$

Theorem 7.

The Voros coefficients have the following form:

$$V_0 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}}+\frac{B_n(\beta_0)}{\beta^{n-1}}+\frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}}+\frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} - \frac{B_n(\gamma_0)+B_n(\gamma_0-1)}{\gamma^{n-1}}\right),$$

$$V_1 = \pm\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}}+\frac{B_n(\beta_0)}{\beta^{n-1}}-\frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}}-\frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} - \frac{B_n(\alpha_0+\beta_0-\gamma_0)+B_n(\alpha_0+\beta_0-\gamma_0+1)}{(\alpha+\beta-\gamma)^{n-1}}\right),$$

$$V_\infty = \pm\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}}-\frac{B_n(\beta_0)}{\beta^{n-1}}-\frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}}+\frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} + \frac{B_n(\alpha_0-\beta_0)+B_n(\alpha_0-\beta_0+1)}{(\alpha-\beta)^{n-1}}\right).$$

Here $B_n(x)$ are the Bernoulli polynomials: $\dfrac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \dfrac{B_n(x)}{n!} t^n$.

Theorem 8.

The Borel sum V_0^3 of V_0 in ω_3 can be computed in the form

$$V_0^3 = \frac{1}{2}\log\left(\frac{\alpha^{\alpha_0 - \frac{1}{2} + \alpha\eta} \beta^{\beta_0 - \frac{1}{2} + \beta\eta} \Gamma(\gamma_0 + \gamma\eta)\Gamma(\gamma_0 - 1 + \gamma\eta)}{2\pi(\alpha - \gamma)^{\alpha_0 - \gamma_0 + \frac{1}{2} + (\alpha - \gamma)\eta} (\beta - \gamma)^{\beta_0 - \gamma_0 - \frac{1}{2} + (\beta - \gamma)\eta} \gamma^{2(\gamma_0 - 1 + \gamma\eta)}} \right.$$
$$\left. \times \frac{\Gamma(\alpha_0 - \gamma_0 + 1 + (\alpha - \gamma)\eta)\Gamma(\beta_0 - \gamma_0 + 1 + (\beta - \gamma)\eta)}{\Gamma(\alpha_0 + \alpha\eta)\Gamma(\beta_0 + \beta\eta)} \right)$$

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$$V_0^3 = \frac{1}{2}\log\left(\frac{\alpha^{\alpha_0-\frac{1}{2}+\alpha\eta}\beta^{\beta_0-\frac{1}{2}+\beta\eta}\Gamma(\gamma_0+\gamma\eta)\Gamma(\gamma_0-1+\gamma\eta)}{2\pi(\alpha-\gamma)^{\alpha_0-\gamma_0+\frac{1}{2}+(\alpha-\gamma)\eta}(\beta-\gamma)^{\beta_0-\gamma_0-\frac{1}{2}+(\beta-\gamma)\eta}\gamma^{2(\gamma_0-1+\gamma\eta)}} \times \frac{\Gamma(\alpha_0-\gamma_0+1+(\alpha-\gamma)\eta)\Gamma(\beta_0-\gamma_0+1+(\beta-\gamma)\eta)}{\Gamma(\alpha_0+\alpha\eta)\Gamma(\beta_0+\beta\eta)}\right)$$

- **What about h_0?**

Here $B_n(x)$ are the Bernoulli polynomials: $\dfrac{te^{xt}}{e^t - 1} = \displaystyle\sum_{n=0}^{\infty} \dfrac{B_n(x)}{n!} t^n.$

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$$\left.\times\frac{\Gamma(\alpha_0-\gamma_0+1+(\alpha-\gamma)\eta)\Gamma(\beta_0-\gamma_0+1+(\beta-\gamma)\eta)}{\Gamma(\alpha_0+\alpha\eta)\Gamma(\beta_0+\beta\eta)}\right)$$

- **What about h_0?**

We define h_0 by

$$\int_{a_0}^{x} S_{\mathrm{odd},\leq 0}\,dx = \left(\operatorname*{Res}_{x=0} S_{\mathrm{odd},\leq 0}\right)\log x + \eta\, h_0 + O(x).$$

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The Borel sum V_0^3 of V_0 in ω_3 can be computed in the form

$$V_0^3 = \frac{1}{2} \log \left(\frac{\alpha^{\alpha_0 - \frac{1}{2} + \alpha\eta} \beta^{\beta_0 - \frac{1}{2} + \beta\eta} \Gamma(\gamma_0 + \gamma\eta) \Gamma(\gamma_0 - 1 + \gamma\eta)}{2\pi (\alpha - \gamma)^{\alpha_0 - \gamma_0 + \frac{1}{2} + (\alpha - \gamma)\eta} (\beta - \gamma)^{\beta_0 - \gamma_0 - \frac{1}{2} + (\beta - \gamma)\eta} \gamma^{2(\gamma_0 - 1 + \gamma\eta)}} \right.$$
$$\left. \times \frac{\Gamma(\alpha_0 - \gamma_0 + 1 + (\alpha - \gamma)\eta) \Gamma(\beta_0 - \gamma_0 + 1 + (\beta - \gamma)\eta)}{\Gamma(\alpha_0 + \alpha\eta) \Gamma(\beta_0 + \beta\eta)} \right)$$

- **What about h_0?**

We define h_0 by

$$\int_{a_0}^{x} S_{\mathrm{odd}, \leq 0} dx = (\operatorname*{Res}_{x=0} S_{\mathrm{odd}, \leq 0}) \log x + \eta h_0 + O(x).$$

Then we have

$$\eta h_0 = \frac{1}{2} \left(\left(\alpha_0 - \frac{1}{2} + \alpha\eta \right) \log \alpha + \left(\beta_0 - \frac{1}{2} + \beta\eta \right) \log \beta + \left(\gamma_0 - \beta_0 - \frac{1}{2} + (\gamma - \alpha)\eta \right) \log(\alpha - \gamma) \right.$$
$$\left. + \left(\gamma_0 - \alpha_0 - \frac{1}{2} + (\gamma - \beta)\eta \right) \log(\beta - \gamma) - 2(\gamma\eta + \gamma_0 - 1) \log \gamma \right).$$

Thus we can obtain the relations between the hypergeometric function and the Borel sums of the WKB solutions.

Theorem 9.

If $(\alpha,\beta,\gamma) \in \omega_3$ and x is sufficiently close to the origin, the following relation holds:

$$F(\alpha_0+\alpha\eta,\beta_0+\beta\eta,\gamma_0+\gamma\eta;x) = Cx^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\Psi_+(x,\eta),$$

where $\Psi_+(x,\eta)$ is the Borel sum of the recessive WKB solution ψ_+ normalized at the turning point a_0 and

$$C = \frac{(\Gamma(\alpha_0-\gamma_0+1+(\alpha-\gamma)\eta)\Gamma(\beta_0-\gamma_0+1+(\beta-\gamma)\eta))^{\frac{1}{2}}\Gamma(\gamma_0+\gamma\eta)}{2\sqrt{\pi}(\Gamma(\alpha_0+\alpha\eta)\Gamma(\beta_0+\beta\eta))^{\frac{1}{2}}}.$$

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If $(\alpha, \beta, \gamma) \in \omega_3$ and x is sufficiently close to the origin, the following relation holds:

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$$C = \frac{(\Gamma(\alpha_0 - \gamma_0 + 1 + (\alpha-\gamma)\eta)\Gamma(\beta_0 - \gamma_0 + 1 + (\beta-\gamma)\eta))^{\frac{1}{2}}\Gamma(\gamma_0 + \gamma\eta)}{2\sqrt{\pi}(\Gamma(\alpha_0 + \alpha\eta)\Gamma(\beta_0 + \beta\eta))^{\frac{1}{2}}}.$$

Remark **The above formula coincides with the formula obtained from the special case (Theorem 6; the case where $\alpha_0 = \beta_0 = 1/2,\ \gamma_0 = 1$) by replacing**

$$\alpha \to \alpha + (\alpha_0 - \frac{1}{2})\eta^{-1},\ \beta \to \beta + (\beta_0 - \frac{1}{2})\eta^{-1},\ \gamma \to \gamma + (\gamma_0 - 1)\eta^{-1}.$$

By Watson's lemma, we have

Theorem 10.

If $(\alpha, \beta, \gamma) \in \omega_3$ and x is sufficiently close to the origin, the following asymptotic formula holds as $\eta \to +\infty$:

$$F(\alpha_0 + \alpha\eta, \beta_0 + \beta\eta, \gamma_0 + \gamma\eta; x) \sim Cx^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\psi_+(x,\eta).$$

Here $C = \dfrac{(\Gamma(\alpha_0 - \gamma_0 + 1 + (\alpha-\gamma)\eta)\Gamma(\beta_0 - \gamma_0 + 1 + (\beta-\gamma)\eta))^{\frac{1}{2}}\Gamma(\gamma_0 + \gamma\eta)}{2\sqrt{\pi}(\Gamma(\alpha_0 + \alpha\eta)\Gamma(\beta_0 + \beta\eta))^{\frac{1}{2}}}$

and

$$\psi_+(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\int_{a_0}^{x} S_{\mathrm{odd}}\,dx\right)$$

is the recessive WKB solution normalized at the turning point a_0.

Summary and concluding remarks

- **We have obtained the relation between the hypergeometric function and the Borel sums of the WKB solutions of the hypergeometric differential equation with a large parameter.**
- **At the same time, we have some formulas which give asymptotic expansions of the hypergeometric function with respect to the large parameter. These formulas assume that the Stokes geometry is non-degenerate, while existing works (cf. NIST Handbook of Mathematical Functions; Olde Daalhuis 2003, 2010; Farid Khwaja-Olde Daalhuis 2014) treat the case where the geometry is degenerate.**
- **Combining our results with connections formulas established by the exact WKB analysis, we obtain the asymptotic formulas as $\eta \to \infty$ for all $x \in \mathbb{C} - \{0, 1\}$.**

Thank you for your attention.