Lifespan of solutions to nonlinear Cauchy problems with small analytic data

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Consider the Cauchy problem

$$\begin{cases} \partial_t^m u = F\left(t, \left\{\partial_t^j \partial_x^\alpha u\right\}_{(j,\alpha)\in I_m}\right), \\ \partial_t^i u(0,x) = \phi_i(x), \quad i = 0, 1, \ldots, m-1 \end{cases} \tag{1}$$

- $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R} \times \mathbb{R}^n$
- $\mathbb{N} = \{0, 1, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$, Ω (open) $\subset \mathbb{R}^n$
- $I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n ; j + |\alpha| \leq m, j < m\}$ (where $m \in \mathbb{N}^*$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$), $N = \# I_m$
- $F(t, X)$ is a function on $\mathbb{R}_t \times \mathscr{U}$ where $X = \{X_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \mathbb{C}^N$ and $\mathscr{U}$ is a convex open neighborhood of $0 \in \mathbb{C}^N$

Definition

(1) A C^∞-function $f(x)$ on Ω is said to be uniformly analytic on Ω if

$$\exists C > 0 \exists h > 0 \text{ s.t. } \forall \alpha \in \mathbb{N}^n, \sup_{x \in \Omega} |\partial_x^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|!.$$

- $\mathcal{A}(\Omega)$ is the totality of uniformly analytic functions on Ω

(2) Let $T > 0$, $I_T = (-T, T)$ and $k \in \mathbb{N}$. A function $u(t,x)$ belongs to $C^k(I_T, \mathcal{A}(\Omega))$ if the following hold for any $j \in \{0, 1, \ldots, k\}$ and $\alpha \in \mathbb{N}^n$:

(i) $\partial_t^j \partial_x^\alpha u(t,x) \in C^0(I_T \times \Omega)$,

(ii) $0 < \forall T_1 < T$, $\exists C_1 = C_{T_1} > 0$ $\exists h_1 > 0$ s.t.

$$\sup_{I_{T_1} \times \Omega} |\partial_t^j \partial_x^\alpha u(t,x)| \leq C_1 {h_1}^{|\alpha|} |\alpha|!.$$

Consider the Cauchy problem

$$\begin{cases} \partial_t^m u = F\big(t, \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big), \\ \partial_t^i u(0,x) = \phi_i(x), \quad i = 0, 1, \ldots, m-1 \end{cases} \tag{2}$$

under the assumptions:

(A_1) $F(t, X)$ is continuous and bounded on $\mathbb{R}_t \times \mathscr{U}$, and holomorphic on $\mathscr{U}$ for any fixed $t \in \mathbb{R}$;
(A_2) $F(t, 0) \equiv 0$ for any $t \in \mathbb{R}$;
(A_3) $\phi_i(x) \in \mathcal{A}(\Omega)$.

$$\begin{cases} \partial_t^m u = F\left(t, \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\right), \\ \partial_t^i u(0,x) = \phi_i(x), \quad i = 0, 1, \ldots, m-1 \end{cases} \tag{3}$$

- (3) has a unique local solution $u(t,x) \in C^m((-\delta,\delta), \mathcal{A}(\Omega))$ $\exists \delta > 0$ (Nagumo 1941, Nirenberg 1972, Nishida 1977, and Walter 1985)
- Lifespan := $\sup\{\text{all } \delta\}$
- D'Ancona and Spagnolo 1991, 1995
  - analytic category for nonlinear hyperbolic equations or systems with initial data $\phi_i(x) = \varepsilon u_i(x)$ ($\varepsilon > 0$)
  - proved that the lifespan T_ε tends to ∞ as $\varepsilon \to 0$
  - when $F(t,X)$ is independent of t, they showed that $1/T_\varepsilon = O((\log\log(1/\varepsilon))^{-1})$ (as $\varepsilon \to 0$)

$$\begin{cases} \partial_t^m u = F\big(t, \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big), \\ \partial_t^i u(0,x) = \phi_i(x), \quad i = 0, 1, \dots, m-1 \end{cases} \tag{4}$$

- Gourdin and Mechab 1999, 2001
  - $F(t, X)$ is independent of t without hyperbolicity assumption and using $\phi_i(x) = u_i(\varepsilon x)$
  - They proved that $1/T_\varepsilon = O(\varepsilon^\sigma)$ (as $\varepsilon \to 0$) for some $\sigma > 0$
- Yamane 2006, 2011, 2012
  - second order case without hyperbolicity assumption
  - Simplest case: initial data $\phi_i(x)$ ($i = 0, 1$) with

$$\sup_{x\in\Omega} |\partial_x^\alpha \phi_i(x)| \le \varepsilon^{i+1+|\alpha|} \quad \text{for any } \alpha \in \mathbb{N}^n.$$

  - proved that $1/T_\varepsilon = O(\varepsilon)$ (as $\varepsilon \to 0$)

Let us look at the situation in the analytic case where $\Omega = (-1/2, 1/2)$ and $0 < \varepsilon < 1$.

Example 1

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + 2u\partial_x u, \\ u(0,x) = \dfrac{\varepsilon}{1-\varepsilon x}, \ \ \partial_t u(0,x) = \dfrac{\sqrt{2}\,\varepsilon^2}{(1-\varepsilon x)^2}. \end{cases} \tag{5}$$

Solution:

$$u(t,x) = \frac{\varepsilon}{1-\sqrt{2}\,\varepsilon t - \varepsilon x},$$

Lifespan: $T_\varepsilon = (2-\varepsilon)/(2\sqrt{2}\,\varepsilon)$. T_ε is of order $1/\varepsilon$ (as $\varepsilon \to 0$).

Example 2

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + \partial_x u + 2u\partial_t u, \\ u(0,x) = \dfrac{\sqrt{\varepsilon}}{1-\varepsilon x}, \quad \partial_t u(0,x) = \dfrac{\varepsilon}{(1-\varepsilon x)^2}. \end{cases} \tag{6}$$

Solution satisfies the relation:

$$\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}\,t-\varepsilon x} \ll u(t,x) \ll \frac{\sqrt{\varepsilon}}{1-(5/2)\sqrt{\varepsilon}\,t-\varepsilon x}$$

Lifespan satisfies the relation:

$$(2-\varepsilon)/(5\sqrt{\varepsilon}) \le T_\varepsilon \le (2-\varepsilon)/(2\sqrt{\varepsilon})$$

T_ε is of order $1/\sqrt{\varepsilon}$ (as $\varepsilon \to 0$).
Note: we write $f(t,x) \ll g(t,x)$ for two formal power series $f(t,x) = \sum_{k,l} a_{k,l} t^k x^l$ and $g(t,x) = \sum_{k,l} b_{k,l} t^k x^l$ if $|a_{k,l}| \le b_{k,l}$ for all (k,l).

Example 3

$$\begin{cases} \partial_t^2 u = 2u\partial_x u, \\ u(0,x) = \dfrac{\varepsilon^3}{1-\varepsilon x}, \quad \partial_t u(0,x) = \dfrac{\varepsilon^5}{(1-\varepsilon x)^2}. \end{cases} \tag{7}$$

Solution:

$$u(t,x) = \frac{\varepsilon^3}{1-\varepsilon^2 t - \varepsilon x}.$$

Lifespan:

$$T_\varepsilon = (2-\varepsilon)/(2\,\varepsilon^2)$$

T_ε is of order $1/\varepsilon^2$ (as $\varepsilon \to 0$).

In the preceding examples, the lifespans of the solutions to the initial value problems (5), (6), and (7) are of orders 1, $1/2$, and 2, respectively, with respect to $1/\varepsilon$.

Problem: *What determines the order of the lifespan of the solution?*

$$F(t,X) = \sum_{(j,\alpha)\in I_m} a_{j,\alpha}(t) X_{j,\alpha} + \sum_{|q|\geq 2} b_q(t) \prod_{(j,\alpha)\in I_m} (X_{j,\alpha})^{q_{j,\alpha}}$$

where $q = \{q_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \mathbb{N}^N$ and $|q| = \sum_{(j,\alpha)\in I_m} q_{j,\alpha}$. We set

$$\Lambda = \{(j,\alpha) \in I_m\,;\, a_{j,\alpha}(t) \not\equiv 0\}, \quad \Delta_2 = \{q \in \mathbb{N}^N\,;\, |q| \geq 2, b_q(t) \not\equiv 0\}.$$

$\Delta_2 \neq \emptyset$ (nonlinear equations)
We define two indices σ and γ by

$$\sigma = \min_{(j,\alpha)\in\Lambda} \left(\frac{|\alpha|}{m-j} \right), \tag{8}$$

$$\gamma \text{ (or } \gamma_\sigma) = \max\left[0,\ \sup_{q\in\Delta_2} \left(\frac{\sigma m - \sigma s_t(q) - s_x(q)}{|q|-1} \right) \right], \tag{9}$$

$$s_t(q) = \sum_{(j,\alpha)\in I_m} j q_{j,\alpha}, \quad s_x(q) = \sum_{(j,\alpha)\in I_m} |\alpha| q_{j,\alpha}. \tag{10}$$

Note that if σ exists then $0 \leq \sigma \leq 1$ and $0 \leq \gamma \leq \sigma m$.

Theorem 1 (The case $\Lambda \neq \emptyset$)

There are $c_i > 0$ $(i = 0, 1, \ldots, m-1)$, $\mu > 0$, and $\varepsilon_0 > 0$ for which the following is true for all $0 < \varepsilon \leq \varepsilon_0$: If the Cauchy data satisfy

$$\sup_{x \in \Omega} |\partial_x^\alpha \phi_i(x)| \leq c_i \varepsilon^{\gamma + \sigma i + |\alpha|} |\alpha|! \quad \forall \alpha \in \mathbb{N}^n, \, i = 0, 1, \ldots, m-1, \tag{11}$$

the Cauchy problem (3) has a unique solution $u(t, x) \in C^m((-T, T), \mathcal{A}(\Omega))$ for $T = \mu/\varepsilon^\sigma$.

Theorem 2 (The case $\Lambda = \emptyset$)

For any $\sigma \geq 0$ there are $c_i > 0$ ($i = 0, 1, \ldots, m-1$), $\mu > 0$, and $\varepsilon_0 > 0$ for which the following is true for all $0 < \varepsilon \leq \varepsilon_0$: If the Cauchy data satisfy

$$\sup_{x \in \Omega} |\partial_x^\alpha \phi_i(x)| \leq c_i \varepsilon^{\gamma_\sigma + \sigma i + |\alpha|} |\alpha|! \quad \text{for any } \alpha \in \mathbb{N}^n,\ i = 0, 1, \ldots, m-1, \tag{12}$$

the Cauchy problem (3) has a unique solution $u(t,x) \in C^m((-T,T), \mathcal{A}(\Omega))$ for $T = \mu/\varepsilon^\sigma$.

Example:
Let $\Omega = (-1/2, 1/2)$ and $0 < \varepsilon < 1$. Consider the equation

$$\partial_t^2 u = 2u\partial_x u.$$

We have $\Lambda = \emptyset$ and $\gamma_\sigma = 2\sigma - 1$ for any $\sigma \geq 1/2$. Consider the IVP with data

$$u(0,x) = \frac{\varepsilon^{2\sigma-1}}{1-\varepsilon x}, \;\; \partial_t u(0,x) = \frac{\varepsilon^{3\sigma-1}}{(1-\varepsilon x)^2}.$$

Solution:

$$u(t,x) = \frac{\varepsilon^{2\sigma-1}}{1-\varepsilon^\sigma t - \varepsilon x}.$$

Lifespan: $T_\varepsilon = (2-\varepsilon)/(2\varepsilon^\sigma)$.

For a C^∞-function $f(x)$ on Ω, we define the formal norm $\|f\|_\rho$ by

$$\|f\|_\rho := \sum_{\alpha \in \mathbb{N}^n} \frac{\sup_{x\in\Omega} |\partial_x^\alpha f(x)|}{|\alpha|!} \rho^{|\alpha|}.$$

Similarly, for a function $u(t,x) \in C^0(I \times \Omega)$ that is of class C^∞ in x, we define the formal norm $\|u(t)\|_\rho$ by

$$\|u(t)\|_\rho := \sum_{\alpha \in \mathbb{N}^n} \frac{\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)|}{|\alpha|!} \rho^{|\alpha|}.$$

Properties:

(1) $f(x) \in \mathcal{A}(\Omega) \iff \|f\|_\rho \ll C/(1-h\rho) \ \exists C > 0 \ \exists h > 0$

(2) $\|\partial_x^\alpha f\|_\rho \ll \partial_\rho^{|\alpha|} \|f\|_\rho$ for any $\alpha \in \mathbb{N}^n$

(3) $\|fg\|_\rho \ll \|f\|_\rho \|g\|_\rho$

Set

$$L(q) = \gamma|q| + \sigma s_t(q) + s_x(q).$$

The function $F(t, X)$ on the RHS of (3) may be written in the form

$$F(t, X) = \sum_{|q|\geq 1, L(q)\geq \gamma+\sigma m} b_q(t) \prod_{(j,\alpha)\in I_m} (X_{j,\alpha})^{q_{j,\alpha}}. \qquad (13)$$

If $\phi(x) \in \mathcal{A}(\Omega)$ satisfies

$$\sup_{x\in\Omega} |\partial_x^\alpha \phi(x)| \leq c\varepsilon^{a+|\alpha|}|\alpha|! \quad \text{for any } \alpha \in \mathbb{N}^n,$$

for some $c > 0$, $\varepsilon > 0$, and $a > 0$, then by setting $\delta = n\varepsilon$, we have

$$\|\phi\|_\rho \ll \frac{c\varepsilon^a}{1 - n\varepsilon\rho} = \frac{(cn^{-a})\delta^a}{1 - \delta\rho}.$$

Theorem 3

Suppose that $F(t, X)$ has the form (13) for some $\sigma \geq 0$ and $\gamma \geq 0$. Then there are $c_i > 0$ ($i = 0, 1, \ldots, m-1$), $\mu > 0$, and $\varepsilon_0 > 0$ for which the following is true for all $0 < \varepsilon \leq \varepsilon_0$: If the Cauchy data satisfy

$$\|\phi_i\|_\rho \ll \frac{c_i \varepsilon^{\gamma+\sigma i}}{1-\varepsilon\rho}, \quad i = 0, 1, \ldots, m-1, \tag{14}$$

the Cauchy problem has a unique solution $u(t, x) \in C^m((-T, T), \mathcal{A}(\Omega))$ for $T = \mu/\varepsilon^\sigma$.

$$\begin{cases} \partial_t^m u = F\left(\{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\right), \\ \partial_t^i u(0,x) = \phi_i(x), \quad i = 0, 1, \ldots, m-1. \end{cases} \tag{15}$$

Theorem 4 (Complex-analytic version)

There are $c_i > 0$ ($i = 0, 1, \ldots, m-1$), $\mu > 0$, and $\varepsilon_0 > 0$ for which the following is true for all $0 < \varepsilon \le \varepsilon_0$: If the Cauchy data satisfy

$$\|\phi_i\|_\rho \ll \frac{c_i \varepsilon^{\gamma+\sigma i}}{1 - \varepsilon\rho}, \quad i = 0, 1, \ldots, m-1, \tag{16}$$

the Cauchy problem has a unique solution $u(t,x) \in \mathcal{O}(B_T, \mathcal{A}_c(U))$ for $T = \mu/\varepsilon^\sigma$.

$$\varphi(X) = \frac{1}{4S} \sum_{n=0}^{\infty} \frac{X^n}{(n+1)^2},$$

where

$$S = \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

1. $\varphi(X)$ defines a holomorphic function on $\{X \in \mathbb{C}; |X| < 1\}$
2. $\varphi(X)^2 \ll \varphi(X)$;
3. For any $0 < \varepsilon < 1$, there is a constant $K_\varepsilon > 0$ such that

$$\frac{1}{1 - \varepsilon X} \ll K_\varepsilon \, \varphi(X).$$

Let $T > 0$ and $R > 0$. We denote by $\mathscr{X}(T, R)$ the set of all functions $u(t, x)$ satisfying:
(i) $\partial_x^\alpha u(t, x) \in C^0((-T, T) \times \Omega)$ $\forall \alpha \in \mathbb{N}^n$ and it is of class C^∞ w.r.t. x;
(ii) $\exists M > 0$ s.t.

$$\|u(t)\|_\rho \ll M\varphi\left(\frac{|t|}{T} + \frac{\rho}{R}\right) \quad \forall t \in (-T, T). \tag{17}$$

$\bullet\mathscr{X}(T, R)$, a subset of $C^0((-T, T), \mathcal{A}(\Omega))$, is a Banach space under the norm

$$\|u\| = \inf\{M\,;\ M \text{ satisfies } (17)\}.$$

$$D_t^{-1}u(t,x) := \int_0^t u(\tau,x)d\tau.$$

$$u(t,x) := v(t,x) + \psi(t,x) \quad \text{with} \quad \psi(t,x) := \sum_{i=0}^{m-1} \frac{\phi_i(x)t^i}{i!}.$$

$$\begin{cases} \partial_t^m v = F\big(t, \{\partial_t^j \partial_x^\alpha (v+\psi)\}_{(j,\alpha)\in I_m}\big), \\ \partial_t^i v(0,x) = 0, \quad i = 0,1,\ldots,m-1. \end{cases}$$

If $w(t,x) := \partial_t^m v(t,x)$ then

$$w(t,x) = F\big(t, \{D_t^{-(m-j)}\partial_x^\alpha w + \partial_t^j \partial_x^\alpha \psi\}_{(j,\alpha)\in I_m}\big) =: \Phi[w]. \quad (18)$$

$$\text{CP (1)} \Longleftrightarrow w = \Phi[w]$$

1. Then we prove that for some closed ball $B(C; T, R)$ of $\mathscr{X}(T, R)$, $\Phi : B(C; T, R) \longrightarrow B(C; T, R)$ is a contraction mapping.
2. Let $C > 0$. We define $B(C; T, R) :=$ $\left\{ u(t, x) \in \mathscr{X}(T, R); \|u(t)\|_\rho \ll \varepsilon^{\gamma+\sigma m}|c|C\varphi\left(\frac{|t|}{T} + \frac{\rho}{R}\right) \forall(-T, T) \right\}$
3. Set $T = \mu/\varepsilon^\sigma$, $R = 1/2\varepsilon$.

Then we can show that ϕ is a contraction from $B(C; T, R)$ to itself.

Define the set $\mathcal{M}$ by:

$$\mathcal{M} = \left\{ q \in \Delta_2 \,;\, \gamma = \frac{\sigma m - \sigma s_t(q) - s_x(q)}{|q| - 1} \right\}.$$

Let $U = \{x \in \mathbb{C} \,;\, |x| < 1/2\}$, and let $c_i > 0$ ($i = 0, 1, \ldots, m-1$), $0 < \varepsilon_0 \le 1$ and $d > 0$ be fixed constants. Consider the Cauchy problem

$$\begin{cases} \partial_t^m u = F\big(\{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m} \big), \\ \partial_t^i u(0,x) = \dfrac{c_i \varepsilon^{\gamma+\sigma i}}{(1-\varepsilon x)^{d+i}}, \quad i = 0, 1, \ldots, m-1, \end{cases} \tag{19}$$

where $\varepsilon > 0$ is regarded as a parameter with $0 < \varepsilon \le \varepsilon_0$.

Proposition 5

Let $0 < \varepsilon \leq 1$. *Suppose* $a_{j,\alpha} > 0\ \forall (j,\alpha) \in \Lambda$, $b_q > 0\ \forall q \in \Delta_2$, *and* $\mathcal{M} \neq \emptyset$. *If* $d > 0$ *satisfies*

$$d \geq \frac{m - s_t(q) - s_x(q)}{|q| - 1} \quad \text{for any } q \in \mathcal{M} \tag{20}$$

and if there are $A > 0$ *and* $h > 0$ *such that*

$$c_i \geq A(d)_i h^i \quad \text{for } i = 0, 1, \ldots, m-1, \tag{21}$$

$$\sum_{q \in \mathcal{M}} b_q A^{|q|-1} h^{s_t(q)} \prod_{(j,\alpha) \in I_m} \{(d)_{j+\alpha}\}^{q_{j,\alpha}} \geq (d)_m h^m, \tag{22}$$

then $T_\varepsilon((19)) \leq (2-\varepsilon)/(2h\varepsilon^\sigma)$.

We can show that

$$u(t,x) \gg w(t,x) := \frac{A\varepsilon^{\gamma}}{(1 - h\varepsilon^{\sigma}t - \varepsilon x)^{d}},$$

which implies that lifespan $T_{\varepsilon} \leq (2-\varepsilon)/(2h\varepsilon^{\sigma})$. Therefore we have

$$\frac{\mu}{\epsilon^{\sigma}} \leq T_{\varepsilon} \leq \frac{\eta}{\epsilon^{\sigma}}$$

for some $\mu > 0$ and $\eta > 0$.

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Thank you for your attention.