Integrable differential-delay equations

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Agenda

- Literature review
- Methods in discrete and differentiable integrable systems
- Delay-differential equations: theory
- Delay-differential equations: analysis and results
Motivation

- Delay-differential equations appear naturally in applications in biology, economics, and engineering.
- Analogues of some Painlevé equations have been obtained as reductions of integrable equations and by applying methods in the analysis of discrete equations.
- Addition laws for elliptic functions are prime examples of delay-differential equations.
- Analysis of these equations blends techniques from the analysis of both discrete and differential integrable systems.
Literature review I

- The first instance of an integrable delay-differential equation was found by Quispel, Capel, and Sahadevan as a Lie symmetry reduction of the discrete KdV equation.
- Grammaticos, Ramani, and Moreira considered bi-Ricatti delay-differential equations and obtained analogues of $P_1 - P_4$ using singularity confinement methods.
- Carstea expressed these bi-Ricatti type equations in Hirota bilinear form.
- Joshi obtained delay-differential equations via direct reductions of the Toda equation; Levi and Winternitz considered symmetry reductions of the Toda equation.
The equations obtained by Quispel et. al., Joshi et. al., and Levi and Winternitz can be expressed as

\[ \bar{u} = F[u(z), u'(z), u(z + \eta), u'(z + \eta)] \]

The equations found by Grammaticos et. al. take the form

\[ \left( u'(z), u(z)^2, u(z), 1 \right) A(z) \left( u'(z + \eta), u(z + \eta)^2, u(z + \eta), 1 \right)^T, \]

where \( A(z) \) is a \( 4 \times 4 \) symmetric matrix with \( z \)-dependent entries

These two distinct kinds of equations require different applications of the singularity confinement criterion.
Painlevé property I

- Painlevé looked for second-order rational ODEs with the property that each solution is single-valued about each movable singularity.
- He obtained 50 equations; all but six of these were integrable in terms of known transcendents.
- The remaining six equations are called Painlevé equations and denoted by $P_I - P_{VI}$.
- These equations possess many special properties, including special solutions (rational, algebraic, special function), representations as compatibilities of linear problems, and Bäcklund transformations.
Painlevé property II

- The Painlevé test may be applied to differential equations (ordinary and partial) to isolate integrability candidates.
- The ansatz
  \[ u = \sum_{n=-p}^{p} a_n z^n, \quad p > 0 \]
  is substituted into the equation, we check that \( p \) is integral and that the resulting recurrence relations are consistent and include a number of constants integration corresponding to the order of the differential equation.
- Autonomizations of the Painlevé equations are solved by (Weierstrass and Jacobi) elliptic functions.
Quispel, Roberts, and Thompson found that many reductions of differential-difference equations fell into the class

\[ x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}, \quad y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})} \]

where

\[ f(y_n) = A_0 Y_n \wedge A_1 Y_n, \quad g(x_{n+1}) = A_0^T X_{n+1} \wedge A_1^T X_{n+1}, \]

and \( X_n = (x^2_n, x_n, 1)^T, \ Y_n = (y^2_n, y_n, 1)^T, \) and \( A_0 \) and \( A_1 \) are constant \( 3 \times 3 \)
QRT map II

- The mapping admits a conserved quantity

\[ K(x, y) = \frac{X_n^T A_0 Y_n}{X_n^T A_1 Y_n} = \text{const.} \]

- The one-parameter biquadratic family of curves
  \( KX_n^T A_1 Y_n = X_n^T A_0 Y_n \) can be parameterized in terms of Jacobi elliptic functions, leading to a multiparameter family of solutions to the QRT map

- Specializations of the QRT map, including the symmetric QRT map (where \( f = g \)) have also been studied

- Deautonomizations of the QRT map lead to discrete analogues of all six Painlevé equations
Delay-differential equations: basic theory

These equations involve derivatives and shifts with respect to the same variable:

\[ F[z; u(z), u(z + \eta_1), \ldots, u(z + \eta_m), u'(z), u'(z + \eta_1), \ldots, u'(z + \eta_m), \]

\[ \ldots, u^{(n)}(z), u^{(n)}(z + \eta_1), \ldots, u^{(n)}(z + \eta_m)] = 0. \]

As with analytic difference equations, these are infinite-dimensional systems: initial data must be specified on an interval of width \( \max\{|\eta_1|, \ldots, |\eta_m|\} \) to propagate the solution forward (or backward).

In practice \( \{\eta_1, \ldots, \eta_m\} \) is a set of integer multiples of \( \eta := \min\{|\eta_1|, \ldots, |\eta_m|\} \); we use the notation

\[ \bar{u} = u(z + \eta), \ u = u(z - \eta), \ \bar{\bar{u}} = u(z + 2\eta), \ {m etc.} \]
A simple linear example

\[ u(z) - u(z - 1) = au'(z). \]

The characteristic equation for this equation corresponds to the substitution \( u(z) = e^{\lambda z} \) (as in the case of linear ODEs) and results in the transcendental relation

\[ 1 - e^{-\lambda} = a\lambda \]

By the decomposition \( \lambda = \text{Re} \lambda + i \text{Im} \lambda \) we find a countable infinity of solutions; by linearity we can combine them to form the solution

\[ u(z) = \sum_{n \in \mathbb{N}} c_ne^{\lambdanz}. \]

However, there is no guarantee that we have obtained the general solution.
The Cauchy problem for delay-differential equations

- \( g(u', \bar{u}) = G[z, u, u', \ldots] \)

- Given initial data \( f(z) \) on an interval of width 
  \( \eta = \max\{|\eta_1|, \ldots, |\eta_m|\} \), we have the problem

  \[
  g(u', u) = G[z, u, u', \ldots], \quad z \in [z_0, z_0 + \eta] \\
  u = f(z), \quad z \in [z_0 - \eta, z_0]
  \]

- This is just an ODE on \([z_0, z_0 + \eta]\) as the function \( G \) is known on this interval

- As in the case of analytic difference equations, we will be interested principally in global analytic (meromorphic) solutions to delay-differential equations
A Painlevé-I type example

Starting from the Kac-van Moerbeke (discrete KdV) equation:

$$\frac{d}{dt} u(x, t) = u(x, t) [u(x + 1, t) - u(x - 1, t)]$$

Quispel, Capel, and Sahadevan found the Painlevé-I type equation

$$\alpha u(z) - \beta u'(z) = u(z) [u(z + 1) - u(z - 1)]$$

The equation possesses a Lax pair inherited from the Kac-van Moerbeke Lax pair and has a continuum limit to Painlevé-I.
Notions of integrability I: existence of explicit solutions

\[ \alpha u - \beta u' = u(\bar{u} - u) \]

Rational solution:

\[ u_1 = \frac{\beta(z + 1 + \delta)(z - 2 + \delta)}{2(z + \delta)(z - 1 + \delta)} \]

Soliton solution:

\[ u_2 = \frac{-\beta k(1 + e^{k(z+1)+\delta})(1 + e^{k(z-2)+\delta})}{2 \sinh k(1 + e^{kz+\delta})(1 + e^{k(z-1)+\delta})} \]

where \( k, \delta \) are arbitrary and when \( \alpha = 0 \).
Notions of integrability II: singularity confinement (discrete Painlevé property)

Suppose that at $z = z_0$ we have the initial data $u = u_0$, $u = O(z - z_0)$, $u' = O(1)$. Then,

\[
\bar{u} = \alpha - \beta \frac{u'}{u} + u = \bar{u} = \alpha - \beta \frac{\bar{u}'}{\bar{u}} + \bar{u} = O(u)
\]

\[
\|u\| = \alpha - \beta \frac{\|u\|'}{\|u\|} + \|u\| = O(1)
\]
Elliptic functions I

Addition laws for Weierstrass and Jacobi elliptic functions provide natural examples of delay-differential equations:

\[
\wp(z \pm \eta; g_2, g_3) = 1 + \frac{1}{4} \left[ \wp'(z; g_2, g_3) \mp \wp'((\eta; g_2, g_3)) \right]^2 - \wp(z; g_2, g_3) - \wp(\eta; g_2, g_3)
\]

\[
\text{sn}(z \pm \eta; k) = \frac{\text{sn}(z; k)\text{sn}'((\eta; k)) \pm \text{sn}'(z; k)\text{sn}(\eta; k)}{1 - k^2 \text{sn}^2(z; k)\text{sn}^2(\eta; k)}
\]
Elliptic functions II

From the addition law for $\text{sn} (z; k)$ we see that the function

$$y(z) = k \text{sn} (\Omega z + c; k) \text{sn} (\Omega \eta; k)$$

satisfies the equations

$$y(z \pm \eta) = \frac{Ay \pm By'}{1 - y^2}, \quad \frac{\bar{y}}{y} = \frac{Ay + By'}{Ay - By'}, \quad \text{and} \quad \bar{y} - y = \frac{2By'}{1 - y^2},$$

where $A = \text{cn}(\Omega \eta; k) \text{dn}(\Omega \eta; k) / [k \text{sn} (\Omega \eta; k)]$ and $B = (k\Omega)^{-1}$.

Deautonomizations of these equations using the singularity confinement criterion lead to some of the known Painlevé delay-differential equations.
QRT analogs I

Consider the function

\[
u(z) = \frac{\alpha + \beta y(z)}{\gamma + \delta y(z)}, \quad y(z) = k \text{sn}(\Omega z + c; k) \text{sn}(\Omega \eta; k), \quad \alpha \delta - \beta \gamma = 1.
\]

Then \(\bar{u}u\) and \(\bar{u} - u\) are rational functions in \(u\) and \(u:\ P(u, u') / Q(u, u')\) and \(R(u, u) / Q(u, u')\), respectively. We search for equations

\[
a(u, u') \bar{u}u + b(u, u')(\bar{u} - u) + c(u, u') = 0,
\]

or \((a, b, c) \cdot (P, R, Q) = 0\), so that \((a, b, c) = (P, R, Q) \wedge (X, Y, Z)\).

This is an analog of the symmetric QRT mapping:

\[
x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}
\]
QRT analogs II

- Some known delay-differential Painlevé type equations are not deautonomizations of this class.
- These include two-point equations identified by Grammaticos, Ramani, and Moreira:

\[
\begin{align*}
    u' + \bar{u}' &= (u - \bar{u})^2 + \alpha(u + \bar{u}) + \frac{\beta}{2\alpha} + \gamma e^{2\alpha z} \\
    (u\bar{u})' &= e^{\omega z}(\alpha u^2 + \beta \bar{u}^2) \\
    u\bar{u}' - u'u &= e^{\omega z}(\alpha u^2\bar{u}^2 + \beta) \\
    u\bar{u}' - u'u &= u^2\bar{u}^2 + \alpha(u + \bar{u})
\end{align*}
\]

- We have shown that the second and third equations, upon autonomization, admit multiparameter families of elliptic function solutions. We expect that the others do as well.


