Integrable differential-delay equations

Bjorn Berntson

Department of Mathematics
University College London

AAGADE, Będlewo, September 2015

Joint work with Rod Halburd

Agenda

- Literature review
- Methods in discrete and differentiable integrable systems
- Delay-differential equations: theory
- Delay-differential equations: analysis and results

Motivation

- Delay-differential equations appear naturally in applications in biology, economics, and engineering
- Analogues of some Painlevé equations have been obtained as reductions of integrable equations and by applying methods in the analysis of discrete equations
- Addition laws for elliptic functions are prime examples of delay-differential equations
- Analysis of these equations blends techniques from the analysis of both discrete and differential integrable systems

Literature review I

- The first instance of an integrable delay-differential equation was found by Quispel, Capel, and Sahadevan as a Lie symmetry reduction of the discrete KdV equation
- Grammaticos, Ramani, and Moreira considered *bi-Ricatti* delay-differential equations and obtained analogues of P_{I}- P_{IV} using singularity confinement methods
- Carstea expressed these bi-Ricatti type equations in Hirota bilinear form
- Joshi obtained delay-differential equations via direct reductions of the Toda equation; Levi and Winternitz considered symmetry reductions of the Toda equation

Literature review II

- The equations obtained by Quispel et. al., Joshi et. al., and Levi and Winternitz can be expressed as

$$\overline{u} = F[u(z), u'(z), u(z+\eta), u'(z+\eta)]$$

- The equations found by Grammaticos et. al. take the form

$$\left(u'(z), u(z)^2, u(z), 1\right) A(z) \left(u'(z+\eta), u(z+\eta)^2, u(z+\eta), 1\right)^T,$$

  where $A(z)$ is a 4×4 symmetric matrix with z-dependent entries
- These two distinct kinds of equations require different applications of the singularity confinement criterion

Painlevé property I

- Painlevé looked for second-order rational ODEs with the property that each solution is single-valued about each movable singularity
- He obtained 50 equations; all but six of these were integrable in terms of known transcendents
- The remaining six equations are called Painlevé equations and denoted by $\mathrm{P_I} - \mathrm{P_{VI}}$
- These equations possess many special properties, including special solutions (rational, algebraic, special function), representations as compatibilities of linear problems, and Bäcklund transformations

Painlevé property II

- The Painlevé test may be applied to differential equations (ordinary and partial) to isolate integrability candidates
- The ansatz

$$u = \sum_{n=-p} a_n z^n, \quad p > 0$$

  is substituted into the equation, we check that p is integral and that the resulting recurrence relations are consistent and include a number of constants integration corresponding to the order of the differential equation
- Autonomizations of the Painlevé equations are solved by (Weierstrass and Jacobi) elliptic functions

QRT map I

- Quispel, Roberts, and Thompson found that many reductions of differential-difference equations fell into the class

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}, \quad y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}$$

where

$$f(y_n) = A_0 Y_n \wedge A_1 Y_n, \quad g(x_{n+1}) = A_0^T X_{n+1} \wedge A_1^T X_{n+1},$$

and $X_n = (x_n^2, x_n, 1)^T$, $Y_n = (y_n^2, y_n, 1)^T$, and A_0 and A_1 are constant 3×3

QRT map II

- The mapping admits a conserved quantity

$$K(x, y) = \frac{X_n^T A_0 Y_n}{X_n^T A_1 Y_n} = \text{const.}$$

- The one-parameter biquadratic family of curves $KX_n^T A_1 Y_n = X_n^T A_0 Y_n$ can be parameterized in terms of Jacobi elliptic functions, leading a to multiparameter family of solutions to the QRT map
- Specializations of the QRT map, including the symmetric QRT map (where $f = g$) have also been studied
- Deautonomizations of the QRT map lead to discrete analogues of all six Painlevé equations

Delay-differential equations: basic theory

These equations involve derivatives and shifts *with respect to the same variable*:

$$F[z; u(z), u(z+\eta_1), \dots, u(z+\eta_m), u'(z), u'(z+\eta_1), \dots, u'(z+\eta_m),$$
$$\dots, u^{(n)}(z), u^{(n)}(z+\eta_1), \dots, u^{(n)}(z+\eta_m)] = 0.$$

As with analytic difference equations, these are infinite-dimensional systems: initial data must be specified on an interval of width $\max\{|\eta_1|, \dots, |\eta_m|\}$ to propagate the solution forward (or backward).

In practice $\{\eta_1, \dots, \eta_m\}$ is a set of integer multiples of $\eta := \min\{|\eta_1|, \dots, |\eta_m|\}$; we use the notation

$$\overline{u} = u(z+\eta), \underline{u} = u(z-\eta), \overline{\overline{u}} = u(z+2\eta), \text{etc.}$$

A simple linear example

$$u(z) - u(z-1) = au'(z).$$

The characteristic equation for this equation corresponds to the substitution $u(z) = e^{\lambda z}$ (as in the case of linear ODEs) and results in the transcendental relation

$$1 - e^{-\lambda} = a\lambda$$

By the decomposition $\lambda = \operatorname{Re}\lambda + \mathrm{i}\operatorname{Im}\lambda$ we find a countable infinity of solutions; by linearity we can combine them to form the solution

$$u(z) = \sum_{n\in\mathbb{N}} c_n e^{\lambda_n z}.$$

However, there is no guarantee that we have obtained the general solution.

The Cauchy problem for delay-differential equations

- $g(u', \overline{u}) = G[z, u, \underline{u}, \underline{u}', \ldots]$
- Given initial data $f(z)$ on an interval of width $\eta = \max\{|\eta_1|, \ldots, |\eta_m|\}$, we have the problem

$$g(u', u) = G[z, \underline{u}, \underline{u}', \ldots], \quad z \in [z_0, z_0 + \eta] \tag{1}$$

$$u = f(z), \quad z \in [z_0 - \eta, z_0] \tag{2}$$

- This is just an ODE on $[z_0, z_0 + \eta]$ as the function G is known on this interval
- As in the case of analytic difference equations, we will be interested principally in global analytic (meromorphic) solutions to delay-differential equations

A Painlevé-I type example

Starting from the Kac-van Moerbeke (discrete KdV) equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x,t) = u(x,t)\left[u(x+1,t) - u(x-1,t)\right]$$

Quispel, Capel, and Sahadevan found the Painlevé-I type equation

$$\alpha u(z) - \beta u'(z) = u(z)\left[u(z+1) - u(z-1)\right]$$

The equation possesses a Lax pair inherited from the Kac-van Moerbeke Lax pair and has a continuum limit to Painlevé-I.

Notions of integrability I: existence of explicit solutions

$$\alpha u - \beta u' = u(\overline{u} - \underline{u})$$

Rational solution:

$$u_1 = \frac{\beta(z+1+\delta)(z-2+\delta)}{2(z+\delta)(z-1+\delta)}$$

Soliton solution:

$$u_2 = \frac{-\beta k(1+e^{k(z+1)+\delta})(1+e^{k(z-2)+\delta})}{2\sinh k(1+e^{kz+\delta})(1+e^{k(z-1)+\delta})}$$

where k, δ are arbitrary and when $\alpha = 0$.

Notions of integrability II: singularity confinement (discrete Painlevé property)

Suppose that at $z = z_0$ we have the initial data $\underline{u} = u_0$, $u = O(z - z_0)$, $u' = O(1)$. Then,

$$\begin{aligned}
\overline{u} =& \alpha - \beta \frac{u'}{u} + \underline{u} = \\
\overline{\overline{u}} =& \alpha - \beta \frac{\overline{u}'}{\overline{u}} + u = \\
\overline{\overline{\overline{u}}} =& \alpha - \beta \frac{\overline{\overline{u}}'}{\overline{\overline{u}}} + \overline{u} = O(u) \\
\overline{\overline{\overline{\overline{u}}}} =& \alpha - \beta \frac{\overline{\overline{\overline{u}}}'}{\overline{\overline{\overline{u}}}} + \overline{\overline{u}} = O(1)
\end{aligned}$$

Elliptic functions I

Addition laws for Weierstrass and Jacobi elliptic functions provide natural examples of delay-differential equations:

$$\wp(z \pm \eta; g_2, g_3) =$$

$$\frac{1}{4}\left[\frac{\wp'(z; g_2, g_3) \mp \wp'(\eta; g_2, g_3)}{\wp(z; g_2, g_3) - \wp(\eta; g_2, g_3)}\right]^2 - \wp(z; g_2, g_3) - \wp(\eta; g_2, g_3)$$

$$\mathrm{sn}\,(z \pm \eta; k) = \frac{\mathrm{sn}\,(z; k)\mathrm{sn}\,'(\eta; k) \pm \mathrm{sn}\,'(z; k)\,\mathrm{sn}\,(\eta; k)}{1 - k^2\,\mathrm{sn}\,^2(z; k)\,\mathrm{sn}\,^2(\eta; k)}$$

Elliptic functions II

From the addition law for $\mathrm{sn}\,(z;k)$ we see that the function

$$y(z) = k\,\mathrm{sn}\,(\Omega z + c; k)\mathrm{sn}\,(\Omega\eta; k)$$

satisfies the equations

$$y(z\pm\eta) = \frac{Ay \pm By'}{1-y^2}, \quad \frac{\overline{y}}{\underline{y}} = \frac{Ay+By'}{Ay-By'} \quad \text{and} \quad \overline{y}-\underline{y} = \frac{2By'}{1-y^2},$$

where $A = \mathrm{cn}(\Omega\eta;k)\,\mathrm{dn}(\Omega\eta;k)/[k\,\mathrm{sn}\,(\Omega\eta;k)]$ and $B = (k\Omega)^{-1}$.

Deautonomizations of these equations using the singularity confinement criterion lead to some of the known Painlevé delay-differential equations.

QRT analogs I

Consider the function

$$u(z) = \frac{\alpha + \beta y(z)}{\gamma + \delta y(z)},\ y(z) = k\,\mathrm{sn}\,(\Omega z + c; k)\mathrm{sn}\,(\Omega\eta; k),\ \alpha\delta - \beta\gamma = 1.$$

Then $\overline{u}\underline{u}$ and $\overline{u} - \underline{u}$ are rational functions in u and u': $P(u, u')\,/\,Q(u, u')$ and $R(u, u)\,/\,Q(u, u')$, respectively. We search for equations

$$a(u, u')\overline{u}\underline{u} + b(u, u')(\overline{u} - \underline{u}) + c(u, u') = 0,$$

or $(a, b, c) \cdot (P, R, Q) = 0$, so that $(a, b, c) = (P, R, Q) \wedge (X, Y, Z)$. This is an analog of the symmetric QRT mapping:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}$$

QRT analogs II

- Some known delay-differential Painlevé type equations are not deautonomizations of this class.
- These include two-point equations identified by Grammaticos, Ramani, and Moreira:

$$
\begin{aligned}
u' + \overline{u}' &= (u - \overline{u})^2 + \alpha(u + \overline{u}) + \frac{\beta}{2\alpha} + \gamma e^{2\alpha z} \\
(u\overline{u})' &= e^{\omega z}(\alpha u^2 + \beta \overline{u}^2) \\
u\overline{u}' - u'\overline{u} &= e^{\omega z}(\alpha u^2 \overline{u}^2 + \beta) \\
u\overline{u}' - u'\overline{u} &= u^2 \overline{u}^2 + \alpha(u + \overline{u})
\end{aligned}
$$

- We have shown that the second and third equations, upon autonomization, admit multiparameter families of elliptic function solutions. We expect that the others do as well.

- B. Grammaticos, A. Ramani and I. C. Moreira, *Delay-differential equations and the Painlevé transcendents*, Phys. A, **196** (1993), 574–590.
- G. R. W. Quispel, H. W. Capel and R. Sahadevan, *Continuous symmetries of differential-difference equations: the Kac-van Moerbeke equation and Painlevé reduction*, Phys. Lett. A **170** (1992), 379–383.
- N. Joshi, *Direct 'delay' reductions of the Toda equation*, J. Phys. A: Math. Theor. **42** (2009), 1–8.
- C. Viallet, *Algebraic entropy for differential delay equations*, arXiv:1408.6161 (2014), 1-7.
- R. G. Halburd and R. Korhonen, *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. Lond. Math. Soc. (3), *94* (2007), 443–474.