Inverse monodromy problems and middle convolution

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September 13-19, 2015
AAGADE, Poland, Bedlewo

<u>Inverse monodromy problems for Fuchsian systems</u>

Fuchsian system

$$\frac{dy}{dz} = \sum_{i=1}^{n} \frac{A_i}{z - a_i}, \quad \sum_{i=1}^{n} A_i = -A_{n+1}$$

with singularities $a_1, \ldots, a_n, a_{n+1} = \infty$.

Monodromy representation

$$\chi : \pi_1(\bar{\mathbb{C}} \setminus \{a_1, \ldots, a_n, a_{n+1}\}, z_0) \to \mathbb{GL}(p, \mathbb{C})$$

$G_i = \chi([\gamma_i])$, $G_1, \ldots, G_n$ are generators of the monodromy group.

The Riemann–Hilbert problem: show that there always exists a linear differential equation of a Fuchsian type (or a Fuchsian system) with given poles and a given monodromy group.

Isomonodromic deformations: include system in the family of Fuchsian systems with constant monodromy.

Additive and multiplicative middle convolution

- N. M. Katz, *Rigid Local Systems*, Annals of Mathematics Studies **139**, Princeton University Press, Princeton, NJ, 1996.

- M. Dettweiler and S. Reiter, *An algorithm of Katz and its application to the inverse Galois problem*, J. Symbolic Comput. **30** (2000), 761–798.

- M. Dettweiler and S. Reiter, *Middle convolution of Fuchsian systems and the construction of rigid differential systems*, Journal of Algebra **318** (2007), 1–24.

- M. Dettweiler and S. Reiter, *Painlevé equations and the middle convolution*, Advances in Geometry **7** (2007), 317–330.

Multiplicative middle convolution (MC_λ)

Multiplicative middle convolution is a transformation of tuples of matrices:

$$MC_\lambda : (G_1, \ldots, G_n) \in (\mathbb{C}^{p\times p})^n \to MC_\lambda(G_1, \ldots, G_n) \in (\mathbb{C}^{m\times m})^n$$

A convolution of the tuple G is a tuple $C_\lambda(G) = (M_1, \ldots, M_n)$ given by

$$M_k = \begin{pmatrix} I & & & & & & \\ & \ddots & & & & & \\ & & I & & & & \\ \lambda(G_1 - I) & \ldots & \lambda(G_{k-1} - I) & \lambda G_k & G_{k+1} - I & \ldots & G_n - I \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{pmatrix},$$

where $k = 1, \ldots, n$ and I is an $(p \times p)$ identity matrix, $M_k \in \mathbb{C}^{pn\times pn}$.

Define

$$\mathcal{K}_k = \begin{pmatrix} 0 \\ \vdots \\ \ker(G_k - I) \\ \vdots \\ 0 \end{pmatrix} \subset \mathbb{C}^{np}, \quad k = 1, \dots, n,$$

$$\mathcal{L} = \cap_{k=1}^{n} \ker(M_k - I) = \ker(M_1 \cdot \ldots \cdot M_n - I),$$

$$\mathcal{K} = \oplus_{i=1}^{n} \mathcal{K}_i.$$

Definition 1 *Fix an isomorphism between $\mathbb{C}^{np}/(\mathcal{K} + \mathcal{L})$ and $\mathbb{C}^m$ for some m. Then a tuple of matrices*

$$MC_\lambda(G) = (\tilde{G}_1, \dots, \tilde{G}_n), \quad \tilde{G}_i \in \mathbb{GL}(m, \mathbb{C}),$$

is called a middle convolution of the tuple G, where $\tilde{G}_k$ is induced by the action of M_k on $\mathbb{C}^m \cong \mathbb{C}^{np}/(\mathcal{K} + \mathcal{L})$.

Additive middle convolution (mc_μ)

Additive middle convolution is a transformation of tuples of matrices:

$$mc_\mu : (A_1, \dots, A_n) \in (\mathbb{C}^{p\times p})^n \to mc_\mu(A_1, \dots, A_n) \in (\mathbb{C}^{m\times m})^n$$

Define

$$B_k = \begin{pmatrix} O & \dots & O & O & O & \dots & O \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ A_1 & \dots & A_{k-1} & A_k + \mu I & A_{k+1} & \dots & A_n \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ O & \dots & O & O & O & \dots & O \end{pmatrix} \in \mathbb{C}^{np\times np},$$

where O denotes a zero $(p \times p)$ matrix. A tuple of matrices $c_\mu(A) = (B_1, \dots, B_n)$ is called a convolution of the tuple A.

Denote

$$l_k = \begin{pmatrix} 0 \\ \vdots \\ \ker(A_k) \\ \vdots \\ 0 \end{pmatrix} \subset \mathbb{C}^{np}, \quad k = 1, \ldots, n,$$

$$\tilde{l} = \oplus_{i=1}^{n} l_k,$$

$$l = \cap_{k=1}^{n} \ker(B_k) = \ker(B_1 + \ldots + B_n).$$

Definition 2 *Fix an isomorphism between* $\mathbb{C}^{np}/(\tilde{l}+l)$ *and* $\mathbb{C}^m$ *for some* m. *Then a tuple of matrices*

$$mc_\mu(A) = (\tilde{A}_1, \ldots, \tilde{A}_n), \quad \tilde{A}_i \in \mathbb{C}^{m\times m}$$

is called a middle convolution, where $\tilde{A}_i$ *is induced by the action of* B_i *on* $\mathbb{C}^m \cong \mathbb{C}^{np}/(\tilde{l}+l)$.

Theorem 1 *(M. Dettweiler and S. Reiter) Let*

$$A = (A_1, \ldots, A_n), \quad A_i \in \mathbb{C}^{p\times p},$$

$$Mon(D_A) = (G_1, \ldots, G_n), \quad \mu \in \mathbb{C}\backslash\mathbb{Z}, \quad \lambda = e^{2\pi\mathbf{i}\mu}.$$

Moreover, assume that the following conditions hold:
1) $p > 1$ *or* $p = 1$ *and at least two elements of* $G_1, \ldots, G_n$ *are not identity matrices;*
2) $G_1, \ldots, G_n$ *generate an irreducible subgroup of* $\mathbb{GL}(p, \mathbb{C})$*;*
3) $\mathsf{rk}(A_i) = \mathsf{rk}(G_i - I)$*;*
4) $\mathsf{rk}(A_1 + \ldots + A_n + \mu I) = \mathsf{rk}(\lambda G_1 \cdot \ldots \cdot G_n - I)$*.*
Then

$$Mon(D_{mc_{\mu-1}(A)}) = MC_\lambda(Mon(D_A))$$

(the fundamental matrix solutions defining monodromy matrices should be chosen in a special way).

$$
\begin{array}{ccc}
Mon(D_A) = (G_1, \dots, G_n) & \xrightarrow{MC_\lambda} & (\tilde{G}_1, \dots, \tilde{G}_n) = Mon(D_{\tilde{A}}) \\
\chi \Big\uparrow \quad \Big\downarrow RH & & \chi \Big\uparrow \quad \Big\downarrow RH \\
A = (A_1, \dots, A_n) & \xrightarrow{mc_{\mu-1}} & (\tilde{A}_1, \dots, \tilde{A}_n) = \tilde{A}
\end{array}
$$

Main idea

We can illustrate main idea by the following diagram, where the question mark means that we want to find a Fuchsian system for a given tuple of monodromy matrices (and singularities), i.e., want to solve the Riemann–Hilbert problem constructively:

$$
\begin{array}{ccc}
Mon(D_A) = (G_1, \dots, G_n) & \begin{array}{c} \xrightarrow{MC_\lambda} \\ \xleftarrow{MC_{\lambda^{-1}}} \end{array} & (\tilde{G}_1, \dots, \tilde{G}_n) = Mon(D_{\tilde{A}}) \\
\chi \Big\uparrow \quad \Big\downarrow RH & & \chi \Big\uparrow \quad \Big\downarrow RH \\
? & \begin{array}{c} \xrightarrow{mc_{\mu-1}} \\ \xleftarrow{mc_{-\mu+1}} \end{array} & (\tilde{A}_1, \dots, \tilde{A}_n) = \tilde{A}.
\end{array}
$$

In other words,

$$
Mon(D_A) = Mon(D_{mc_{-\mu+1}(\tilde{A})}) = MC_{\lambda^{-1}}(Mon(D_{\tilde{A}})) = MC_{\lambda^{-1}}(\tilde{G}) =
$$

$$
= MC_{\lambda^{-1}}(MC_\lambda(G)) = (CG_1C^{-1}, \dots, CG_nC^{-1}), \quad C \in \mathbb{GL}(p, \mathbb{C}).
$$

Constructive solutions

Methods using isomonodromic deformations

- Ph. Boalch, *Some explicit solutions to the Riemann–Hilbert problem*, in *Differential Equations and Quantum Groups*, IRMA Lect. Math. Theor. Phys. **9**, Eur. Math. Soc., Zurich (2007), 85–112.

- D. Korotkin, *Solution of matrix Riemann–Hilbert problems with quasi-permutation monodromy matrices*, Math. Ann. **329** (2004), 335–364.

Methods using matrix series

- Lappo–Danilevsky

- Erugin

- Krylov

The R.–H. problem: constructive solutions of Lappo-Danilevsky

Monodromy matrices $G_1, \ldots, G_n$, singular points $a_1, \ldots, a_n$.

$$\frac{dy}{dz} = \sum_{i=1}^{n} \frac{A_i}{z - a_i} y$$

$$A_i = \frac{1}{2\pi\mathbf{i}} \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,n} Q_i(a_{j_1}, \ldots, a_{j_\nu} | b)(G_{j_1} - I) \cdot \ldots \cdot (G_{j_\nu} - I),$$

where the coefficients $Q_i(a_{j_1}, \ldots, a_{j_\nu} | b)$ can be found recursively. The series converges when every $\|G_i - I\|$ is small. Here the indices $j_1, \ldots, j_\nu$ take values $1, \ldots, n$ independently in the sum $\sum_{j_1,\ldots,j_\nu}^{1,\ldots,n}$.

The R.–H. problem: constructive solutions of Lappo-Danilevsky

- J. A. Lappo–Danilevskij (J. A. Lappo–Danilevsky), *Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. I*, Travaux Inst. Physico-Math. Stekloff, **6**, Acad. Sci. USSR, Leningrad, 1934, 1256.

- J. A. Lappo–Danilevskij (J. A. Lappo–Danilevsky), *Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. II*, Travaux Inst. Physico-Math. Stekloff, **7**, Acad. Sci. USSR, Moscow–Leningrad, 1935, 5210.

- J. A. Lappo–Danilevskij (J. A. Lappo–Danilevsky), *Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. III*, Travaux Inst. Physico-Math. Stekloff, **8**, Acad. Sci. USSR, Moscow–Leningrad, 1936, 5206.

The R.–H. problem: constructive solutions of Erugin

Assume that the 2×2 matrices W_1, W_2, W_3 and the singularities $a_1, a_2, a_3, a_4 = \infty$ are given. Then find a 2×2 linear differential system

$$(1) \quad \frac{dy}{dz} = \left(\frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} + \frac{A_3}{z - a_3} \right) y, \qquad A_1 + A_2 + A_3 = -A_4,$$

such that the fundamental matrix solution of the system $Y(z)$ normalized by $Y(b) = I$ is given by $Y(z) = \bar{Y}_i(z)(z - a_i)^{W_i}$ in the neighborhood of each singularity a_i. Here $\bar{Y}_i(z)$ is a second order matrix homomorphic in the neighborhood of a_i. Furthermore, Erugin studied properties of A_i as the function of W_i.

The R.–H. problem: constructive solutions of Erugin

- N. P. Erugin, *The Riemann Problem*, Nauka i Technika, Minsk, 1982 (in Russian).

- N. P. Erugin, *The Riemann problem. I*, Differencial'nye Uravnenija **11** (1975), 771–781 (in Russian).

- N. P. Erugin, *The Riemann problem. II*, Differencial'nye Uravnenija **12** (1976), 779–799 (in Russian).

- N. P. Erugin, *The Riemann problem. III. The case $n = 2$ and $m = 4$*, Differencial'nye Uravnenija **13** (1977), 238–254 (in Russian).

- N. P. Erugin, *The Riemann problem*, Differential Equations **25** (1989), 907–911.

The R.–H. problem: Krylov's method for the Gauss system

Lappo–Danilevsky also considered a particular case of a Fuchsian (2×2) system with three singularities

$$\text{(2)} \qquad \frac{dy}{dz} = \left(\frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} \right) y, \quad A_1 + A_2 = -A_3.$$

System (2) is called the Gauss system. Lappo–Danilevsky fully solved the Riemann–Hilbert problem in terms of the matrix series for arbitrary monodromy matrices (not only for matrices lying in the neighborhood of the identity matrix). Later on Krylov substantially improved this result and described the solutions in terms of the hypergeometric series and studied their multivaluedness.

- B. L. Krylov, *Explicit solution of Riemann problem for Gauss system*, Tr. Kazan Av. Inst. **31** (1956), 203–445 (in Russian).

New theorem on the R.–H. problem constructive solutions

Theorem 2 *Let* $a_1, a_2, a_3, a_4 = \infty$ *be four singular points and assume that the matrices* G_1, G_2, G_3, G_4, $G_k \in \mathbb{GL}(p, \mathbb{C})$, $p > 2$, *with* $G_1 \cdot \ldots \cdot G_4 = I$, *satisfy the following conditions:*
1) the tuple $G = (G_1, G_2, G_3)$ *is irreducible;*
2) there exists $\lambda \in \mathbb{C} \backslash \{0, 1\}$, *such that* $\dim MC_\lambda(G) = 2.^{*)}$

Then for a given tuple G *of monodromy matrices there exists a constructive solution (constructive algorithm) to the Riemann–Hilbert problem.*

$*)$ If $\lambda \neq 1$, $\lambda \neq 0$, then

$$\dim(MC_\lambda(G)) = \sum_{k=1}^{n} \mathrm{rk}(G_k - I) - p + \mathrm{rk}(\lambda G_1 \cdot \ldots \cdot G_n - I).$$

Example 1

$$G_1 = \begin{pmatrix} \mathbf{i} & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2\mathbf{i} & 2\mathbf{i} & 0 \\ 2\mathbf{i} & -\mathbf{i} & -2\mathbf{i} \end{pmatrix}.$$

The tuple (G_1, G_2) is irreducible. We apply $MC_{\mathbf{i}}$.

$$\tilde{G}_1 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \tilde{G}_2 = \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix}.$$

We can get residue matrices according to Krylov's method or using well known reasons.

Some explanations on construction

Every pair of (2×2)-matrices $\tilde{A}_i, \tilde{A}_h$ can be transformed (conjugating by a constant non-degenerate matrix C) to one of the four types

$$(I) \qquad C\tilde{A}_iC^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 1 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_hC^{-1} = \begin{pmatrix} \chi_h^{(1)} & \nabla \\ 0 & \chi_h^{(2)} \end{pmatrix};$$

$$(II) \qquad C\tilde{A}_iC^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 1 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_hC^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ 0 & \chi_h^{(2)} \end{pmatrix};$$

$$(III) \qquad C\tilde{A}_iC^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 0 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_hC^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ 0 & \chi_h^{(2)} \end{pmatrix};$$

$$(IV) \qquad C\tilde{A}_iC^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 1 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_hC^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ \nabla & \chi_h^{(2)} \end{pmatrix};$$

where $i \neq h, i, h \in \{1, 2\}$, $\nabla \in \mathbb{C}$, $\nabla \neq 0$, $\nabla + (\chi_1^{(1)} - \chi_1^{(2)})(\chi_2^{(1)} - \chi_2^{(2)}) \neq 0$.

$\eta_i^{(1)}, \eta_i^{(2)}$ are eigenvalues of monodromy matrix $\tilde{G}_i$.

$\chi_i^{(1)}, \chi_i^{(2)}$ are eigenvalues of residue matrix $\tilde{A}_i$.

$$\chi_i^{(k)} = \frac{1}{2\pi\mathbf{i}}\mathrm{Log}(\eta_i^{(k)}), \quad \mathbf{i} = \sqrt{-1},$$

brunches of logarithm should be chosen so that eigenvalues satisfy relation

$$\chi_1^{(1)} + \chi_1^{(2)} + \chi_2^{(1)} + \chi_2^{(2)} + \chi_3^{(1)} + \chi_3^{(2)} = 0.$$

(I)

$$C\tilde{A}_iC^{-1} = \begin{pmatrix} \chi_i^{(1)} & \nabla \\ 0 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_hC^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ 1 & \chi_h^{(2)} \end{pmatrix}, \quad i \neq h, i, h \in \{1, 2\},$$

where

$$\nabla = (\chi_1^{(1)} + \chi_2^{(1)})(\chi_1^{(2)} + \chi_2^{(2)}) - \chi_3^{(1)}\chi_3^{(2)}.$$

Indeed

$$\tilde{A}_3 = -(\tilde{A}_1 + \tilde{A}_2) = -\begin{pmatrix} \chi_1^{(1)} + \chi_2^{(1)} & \nabla \\ 1 & \chi_1^{(2)} + \chi_2^{(2)} \end{pmatrix},$$

$$\det \tilde{A}_3 = \chi_3^{(1)}\chi_3^{(2)} = (\chi_1^{(1)} + \chi_2^{(1)})(\chi_1^{(2)} + \chi_2^{(2)}) - \nabla.$$

$$\tilde{A}_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}, \tilde{A}_2 = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\pi \mathbf{i}}\mathrm{Log}(2) & \frac{1}{9} \\ 0 & \frac{1}{2\pi \mathbf{i}}\mathrm{Log}(2) \end{pmatrix},$$

$$\tilde{A}_3 = -\tilde{A}_1 - \tilde{A}_2 = \begin{pmatrix} -\frac{1}{2\pi \mathbf{i}}\mathrm{Log}(2) & -\frac{1}{9} \\ -1 & -\frac{1}{2\pi \mathbf{i}}\mathrm{Log}(2) \end{pmatrix}.$$

System for monodromy $\tilde{G}_1, \tilde{G}_2$

$$\frac{dy}{dz} = \frac{\tilde{A}_1}{z - a_1} + \frac{\tilde{A}_2}{z - a_2}, \quad a_1 \neq a_2,$$

has only non-resonant singular points.

Solution of the Riemann-Hilbert problem is not unique.

We apply $mc_{\frac{3}{4}}$.

Resulting system for G_1, G_2:

$$\frac{dy}{dz} = \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2}.$$

$$A_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} + \frac{\log(2)}{2\pi \mathbf{i}} & \frac{1}{9} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{5}{4} + \frac{\log(2)}{2\pi \mathbf{i}} & \frac{1}{9} \\ 1 & 0 & \frac{3}{4} + \frac{\log(2)}{2\pi \mathbf{i}} \end{pmatrix}.$$

Example 2

$$G_1 = \begin{pmatrix} a & -2\pi\mathbf{i}\tau & 2\pi\mathbf{i}\tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2\pi\mathbf{i}\kappa & a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\pi\mathbf{i}\kappa & 0 & a \end{pmatrix}, \qquad \tau, \kappa \in (0,1),\ a \in \mathbb{C}\backslash\{0,1\}.$$

We apply $MC_{a^{-1}}$ and get a tuple of 2×2 matrices

$$\tilde{G}_1 = \begin{pmatrix} 1 & 0 \\ -2\pi\mathbf{i}\kappa & 1 \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} 1 & -2\pi\mathbf{i}\tau \\ 0 & 1 \end{pmatrix}, \quad \tilde{G}_3 = \begin{pmatrix} 1 & 2\pi\mathbf{i}\tau \\ 0 & 1 \end{pmatrix}.$$

$$\frac{dy}{dz} = \left(\sum_{i=1}^{3} \frac{\tilde{A}_i}{z - a_i}\right) y, \quad \tilde{A}_i = \begin{pmatrix} -\sqrt{\frac{-\sigma_{i1}\sigma_{i2}}{\tau\kappa}} & \frac{\sigma_{i2}}{-\tau} \\ \frac{\sigma_{i1}}{-\kappa} & \sqrt{\frac{-\sigma_{i1}\sigma_{i2}}{\tau\kappa}} \end{pmatrix},$$

where the values σ_{i1}, σ_{i2} are the sums of the power series in $t = \frac{a_3 - a_1}{a_3 - a_2}$, $|t| < 1$.

- V. V. Amelkin and M. N. Vasilevich, *Construction of the second-order Fuchsian systems with nilpotent irreducible residue matrices*, Scientific Publications of the State University of Novi Pazar, Ser. A: Appl. Math. Inform. and Mech. **5** (1) (2013), 7–15.

In particular,

$$\sigma_{ik} = \mathrm{tr}(\tilde{A}_i W_k),$$

where

$$W_k = \tilde{A}_k + \mathrm{tr}(\tilde{A}_i \tilde{A}_k) + \sum_{\nu=2}^{\infty} \sum_{j_1,\dots,j_\nu}^{1,2,3} P_k^*(a_{j_1}, \dots, a_{j_\nu} | b) \mathrm{tr}(\tilde{A}_{j_1} \cdot \dots \cdot \tilde{A}_{j_\nu})$$

and

$$P_k^*(a_{j_1}, \dots, a_{j_\nu} | b) = \frac{1}{2\pi \mathbf{i}} P_k(a_{j_1}, \dots, a_{j_\nu} | b),$$

$$P_k(a_{j_1}|b) = \begin{cases} 2\pi\mathbf{i}, & k = j_1, \\ 0, & k \neq j_1, \end{cases}$$

$$P_k(a_{j_1}, \ldots, a_{j_\nu}|b) = \frac{(2\pi\mathbf{i})^\nu}{\nu!}, \quad j_1 = \ldots = j_\nu = k,$$

$$P_k(a_{j_1}, \ldots, a_{j_\nu}|b) = \int_{a_k}^{b} \left(\frac{P_k(a_{j_1}, \ldots, a_{j_{\nu-1}}|b)}{b - a_{j_\nu}} - \frac{P_k(a_{j_2}, \ldots, a_{j_\nu}|b)}{b - a_{j_1}} \right) db.$$

We also have

$$\tilde{A}_j^2 = 0, \quad \tilde{A}_j \tilde{A}_k = \rho_{jk} I - \tilde{A}_k \tilde{A}_j, \quad \tilde{A}_j \tilde{A}_k \tilde{A}_j = \tilde{A}_j \rho_{jk},$$

where $\rho_{jk} = \mathrm{tr}(\tilde{A}_j \tilde{A}_k)$.

So $\mathrm{tr}(\tilde{A}_{j_1} \cdot \ldots \cdot \tilde{A}_{j_\nu})$ has expressions in terms of $\rho_{12}, \rho_{13}, \rho_{23}$ and $\rho_{123} = \mathrm{tr}(\tilde{A}_1 \tilde{A}_2 \tilde{A}_3)$.

Values $\rho_{12}, \rho_{13}, \rho_{23}, \rho_{123}, \rho_{132}$ can be found as solutions of system of ODE's.

$$\frac{d\rho_{12}}{dt} = \frac{2\rho_{123}}{t}, \quad \frac{d\rho_{13}}{dt} = \frac{2\rho_{123}}{1-t}, \quad \frac{d\rho_{23}}{dt} = \frac{2\rho_{123}}{t(t-1)},$$

$$\frac{d\rho_{123}}{dt} = \frac{\rho_{13}(\rho_{12} - \rho_{23})}{t} + \frac{\rho_{12}(\rho_{23} - \rho_{13})}{t-1}.$$

$$\rho_{12} = -C_0 + \sum_{n=1}^{+\infty}\Big(\frac{n-1}{n}C_{n-1} - C_n\Big)t^n,$$

$$\rho_{13} = -\sum_{n=2}^{+\infty}\frac{n-1}{n}C_{n-1}t^n,$$

$$\rho_{23} = \sum_{n=0}^{+\infty}C_n t^n,$$

$$\rho_{123} = \frac{1}{2}\sum_{n=1}^{+\infty}\Big((n-1)C_{n-1} - nC_n\Big)t^n, \quad t = \frac{a_3 - a_1}{a_2 - a_1}, \quad |t| < 1.$$

The coefficients $C_0, C_1, \ldots$ are given by

$$C_n = \frac{1}{n^2}\Bigg[(n-1)\Bigg((2n-1)C_{n-1} - (n-2)C_{n-2} + 2C_0(2\frac{n-2}{n-1}C_{n-2} - C_{n-1}) +$$
$$C_1(2\frac{n-3}{n-2}C_{n-3} - C_{n-2} + \ldots + C_{n-3}(2\frac{1}{2}C_1 - C_2) - C_{n-2}C_1 - C_{n-1}C_0)\Bigg) -$$
$$\Bigg(\frac{1}{2}C_1(2C_{n-2} - \frac{n-3}{n-2}C_{n-3}) + \ldots + 2\frac{n-2}{n-1}C_{n-2}C_1 + 2\frac{n-1}{n}C_0\Bigg)\Bigg],$$

$$n = 1, 2, \ldots, \quad C_0 \neq 0.$$

We use $mc_{-\mu+1}$ with the parameter $\mu = \frac{1}{2\pi\mathbf{i}} \ln a^{-1}$.

$$\frac{dy}{dz} = \left(\sum_{i=1}^{3} \frac{A_i}{z - a_i} \right) y,$$

$$A_1 = \begin{pmatrix} -\mu + 1 & \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}}\frac{\sigma_{21}}{\sigma_{11}} - \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}} & \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}}\frac{\sigma_{31}}{\sigma_{11}} - \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}}\frac{\sigma_{11}}{\sigma_{21}} - \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}} & -\mu + 1 & \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}}\frac{\sigma_{31}}{\sigma_{21}} - \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}} \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}}\frac{\sigma_{11}}{\sigma_{31}} - \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}} & \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}}\frac{\sigma_{21}}{\sigma_{31}} - \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}} & -\mu + 1 \end{pmatrix}.$$

Non-Schlesinger isomonodromic deformations

Fuchsian system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{A_i^0}{z - a_i^0} \right) y, \quad \sum_{i=1}^{n} A_i^0 = -A_{n+1}^0$$

with monodromy representation χ^0.

Fuchsian family

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{A_i(a)}{z - a_i} \right) y, \quad \sum_{i=1}^{n} A_i(a) = -A_{n+1}(a), \quad A_i(a^0) = A_i^0,$$

with monodromy representation χ_a.

Definition 3 *The family of Fuchsian systems D_A is called isomonodromic if the monodromy representation χ_a coincides with the monodromy representation χ^0 of the initial system D_{A^0} for any $a \in D(a^0) \setminus \cup_{i,j=1, i \neq j}^{n} \{a_i = a_j\}$.*

Theorem 3 *The family of Fuchsian systems D_A is isomonodromic if and only if there exists a matrix-valued differential 1-form ω on $\mathbb{C} \times D(a^0) \setminus \cup_{i=1}^n \{z - a_i = 0\}$ such that*
i). $\omega = \sum_{i=1}^n \frac{A_i(a)}{z-a_i} dz$ for any fixed $a \in D(a^0)$;
ii). $d\omega = \omega \wedge \omega$.

Theorem 4 *Any matrix-valued differential 1-form ω on $\bar{\mathbb{C}} \times D(a^0) \setminus \cup_{i=1}^n \{z - a_i = 0\}$ which defines isomonodromic deformation of D_A is given by*

$$\omega = \sum_{i=1}^n \frac{A_i(a)}{z - a_i} d(z - a_i) + \sum_{k=1}^n \gamma_k(a) da_k + \sum_{l=1}^n \sum_{k=1}^n \sum_{m=1}^{r_l} \frac{\gamma_{m,k,l}(a)}{(z - a_l)^m} da_k, \tag{3}$$

where $\gamma_{m,k,l}(a), \gamma_k(a)$ are holomorphic in $D(a^0)$ and r_l is a maximal l-resonance of system D_A for $a = a^0$.

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In the Haraoka–Filipuk theorem it was shown that the Schlesinger deformation equations for deformation of non-resonant Fuchsian system are preserved by middle convolution.

We construct two explicit examples. First one shows that Bolibruch's non-Schlesinger deformations of resonant Fuchsian systems may be preserved by middle convolution The second one shows that middle convolution does not in general preserve non-Schlesinger isomonodromic deformations.

Example 3

$$\omega = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{2a}{a^2-1} & -\frac{1}{2} \end{pmatrix} \frac{d(z+a)}{z+a} + \begin{pmatrix} 0 & -6a \\ 0 & -\frac{1}{2} \end{pmatrix} \frac{dz}{z} +$$
$$\begin{pmatrix} 1 & 3a+3 \\ \frac{1}{1+a} & 3 \end{pmatrix} \frac{d(z-1)}{z-1} + \begin{pmatrix} -3/2 & 3a-3 \\ \frac{1}{a-1} & -2 \end{pmatrix} \frac{d(z+1)}{z+1} +$$
$$\begin{pmatrix} 0 & 0 \\ \frac{2a}{a^2-1} & 0 \end{pmatrix} \frac{da}{z+a}.$$

After mc.

$$\frac{dy}{dz} = \left(\frac{A_1(a)}{z+a} + \frac{A_2(a)}{z} + \frac{A_3(a)}{z-1} + \frac{A_4(a)}{z+1} \right) y,$$

$$A_1(a) = \begin{pmatrix} \mu + 1/2 & 0 & -6a & 3a+3 & 3a-3 \\ -\frac{2a}{a^2-1} & \mu - \frac{1}{2} & -\frac{1}{2} & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{2a}{-1+a^2} & -\frac{1}{2} & \mu - \frac{1}{2} & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{11a+1}{6(a^2-1)} & -\frac{1}{2} & -\frac{5a+1}{2(a+1)} & \mu + 4 & -\frac{a+3}{a+1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{9a+1}{4(a^2-1)} & -\frac{1}{2} & \frac{5a+1}{2(a-1)} & \frac{3(a-3)}{2(a-1)} & \mu - \frac{7}{2} \end{pmatrix}.$$

This Fuchsian family with five singular points is isomonodromic since it is defined by the differential 1-form which satisfies conditions of the theorem about dif.form. This form is given by

$$\omega = A_1(a)\frac{d(z+a)}{z+a} + A_2(a)\frac{dz}{z} + A_3(a)\frac{d(z-1)}{z-1} + A_4(a)\frac{d(z+1)}{z+1}$$

$$+ \begin{pmatrix} 0 & 0 & 6 & -3 & -3 \\ \frac{2(2a^2+5a+1)}{(a^2-1)^2} & \frac{2a^2-5a+1}{2(a^2-1)a} & -\frac{47a^2+1}{2(a^2-1)a} & \frac{9a+3}{a^2-1} & \frac{2(7a+1)}{a^2-1} \\ 0 & \frac{1}{2a} & -\frac{1+5a}{2a-2a^3} & 0 & 0 \\ -\frac{1}{6(1+a)^2} & \frac{1}{2(a+1)} & 0 & \frac{3}{a^2-1} & 0 \\ \frac{1}{4(a-1)^2} & \frac{1}{2(a-1)} & 0 & 0 & \frac{2}{a^2-1} \end{pmatrix} da$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{2(2\mu+1)a}{a^2-1} & 0 & \frac{24a^2}{a^2-1} & -\frac{12a}{a-1} & -\frac{12a}{a-1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{da}{z+a}.$$

Publications

- Y. Bibilo and G. Filipuk, *Constructive solutions to the Riemann–Hilbert problem and middle convolution*, submitted.

- Yu. Bibilo, G. Filipuk. *Middle convolution and non-Schlesinger deformations*. Proc. Japan Acad. Ser. A **91**, pp. 66–69 (2015)

- Yu. Bibilo, G. Filipuk. *Non-Schlesinger isomonodromic deformations of Fuchsian systems and middle convolution*. SIGMA **11**, Paper 023, 14pp. (2015)