Value distribution and growth of the second order ODE's

AAGADE, September 2015

Ewa Ciechanowicz (joint work with Galina Filipuk)

Complex differential equations have been analysed from Nevanlinna theory point of view since the beginning of the theory itself in the 1920's. As early as 1933 K. Yosida proved the theorem of J. Malmquist applying methods of Nevanlinna theory. The first systematic application of the theory to solutions of differential equations was conducted in 1940's by H. Wittich and this global approach gained popularity in the 1970's.

1. Basics of Nevanlinna theory

Properties of meromorphic functions are described by certain real functions of $r \in (0, +\infty)$. We have:

a function counting a-points:

$$N(r, a, f) = \int_0^r \{n(t, a, f) - n(0, a, f)\} \frac{dt}{t} - n(0, a, f) \log r,$$

$n(r, a, f)$ - the number of $a-$points of f
(counting multiplicity) in $|z| \leq r$,
$n(0, a, f)$ - multiplicity of an a-point at zero;

a mean proximity (compensating) function:

$$m(r,a,f) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta & \text{for } a = \infty \\ \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{1}{f(re^{i\theta})-a}| d\theta & \text{for } a \neq \infty \end{cases};$$

Nevanlinna characteristic:

$$T(r,f) = m(r,\infty,f) + N(r,\infty,f).$$

$$f \text{ is rational} \qquad \text{iff} \qquad T(r,f) = O(\log r)\ (r \to \infty)$$

Theorem 1. [15] *For any function f meromorphic in the disc $|z| < R \leq \infty$ the equality*

(1) $$m(r,a,f) + N(r,a,f) = T(r,f) + O(1),$$

holds for each $a \in \overline{\mathbb{C}}$.

Theorem 2. [15] *Let f be a meromorphic function in $\mathbb{C}$, $\{a_k\}_{k=1}^q \in \overline{\mathbb{C}}$. Then*

(2) $$\sum_{k=1}^q m(r,a_k,f) \leq 2T(r,f) + O(\log(rT(r,f))),$$

for $r \to \infty$, possibly outside a set $E \subset [0,\infty)$ of finite measure.

A defect of f at a:

$$\delta(a,f) := \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}$$

Defective value: $\delta(a,f) > 0$

From Theorems 1 and 2:

$$0 \le \delta(a,f) \le 1,$$

$$\sum_{a\in\overline{\mathbb{C}}} \delta(a,f) \le 2.$$

Multiplicity index:

$$\vartheta(a,f) = \liminf_{r\to\infty} \frac{N(r,a,f) - \overline{N}(r,a,f)}{T(r,f)}$$

$\overline{N}(r,a,f)$ counts each $a-$point once.

The order and lower order of growth:

$$\varrho(f) := \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}, \qquad \lambda(f) := \liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

$\varrho(f) = \lambda(f)$ - *a function of regular growth*

- Rational functions are of zero order;
- $\varrho(e^z) = \lambda(e^z) = 1$;
- $\varrho(e^{e^z}) = \lambda(e^{e^z}) = \infty$.

2. More information about exceptional values

$$\mathcal{L}(r,a,f) := \begin{cases} \max\limits_{|z|=r} \log^+ |f(z)| & \text{for } a = \infty, \\ \max\limits_{|z|=r} \log^+ \left|\frac{1}{f(z)-a}\right| & \text{for } a \neq \infty. \end{cases}$$

deviation with respect to $a \in \overline{\mathbb{C}}$:

$$\beta(a,f) = \liminf_{r\to\infty}\frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

From definitions:

$$\delta(a,f) \le \beta(a,f)$$

$$E_N(f) : \{a \in \overline{\mathbb{C}} : \ \delta(a,f) > 0\}$$

$$E_\Pi(f) : \{a \in \overline{\mathbb{C}} : \ \beta(a,f) > 0\}.$$

$$E_N(f) \subset E_\Pi(f),$$

Petrenko in 1978:

$$\beta(a,f) \le B(\lambda) := \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \text{if } \ \lambda \le 0.5\,, \\ \pi\lambda & \text{if } \ \lambda > 0.5\,. \end{cases}$$

Marchenko and Shcherba in 1990:

$$\sum_{a\in\overline{\mathbb{C}}} \beta(a,f) \le 2B(\lambda).$$

For $f(z) = \exp(z)$:

- $E_N(f) = E_\Pi(f) = \{0, \infty\}$,
- $\delta(0, f) = \delta(\infty, f) = 1$,
- $\beta(0, f) = \beta(\infty, f) = \pi$,
- $\sum_{a \in \overline{\mathbb{C}}} \delta(a, f) = 2$,
- $\sum_{a \in \overline{\mathbb{C}}} \beta(a, f) = 2\pi$.

Gol'dberg, Eremenko and Sodin in 1987:

Theorem 3. *For any positive number ϱ and any two sets $E_1 \subset E_2 \subset \overline{\mathbb{C}}$, which are at most countable there is a meromorphic function of order ϱ such that*

$$E_N(f) = E_1, \qquad E_\Pi(f) = E_2.$$

(!) Possible to find functions of regular growth with $E_N(f) \neq E_\Pi(f)$.

3. Some applications

We say that $\phi : (0, +\infty) \to \mathbb{R}$ is a $S(r, f)$ if

$$\phi(r) = o(T(r, f)) \qquad (r \to \infty,\ r \notin E)$$

where E is a set of finite linear measure.

A meromorphic function g is small with respect to f if $T(r, a) = S(r, f)$.

Theorem 4. [Lemma on the logarithmic derivative]
Let f be a transcendental meromorphic function, $k \in \mathbb{N}$. Then $m(r, \frac{f^{(k)}}{f}) = S(r, f)$. If f is of finite order of growth, $m(r, \frac{f^{(k)}}{f}) = O(\log r)$.

Theorem 5. [Clunie lemma]
Let f be a transcendental meromorphic solution of

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$, $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients small with respect to f. If the total degree d of $Q(z,f)$ as a polynomial in f and its derivatives is $d \leq n$ then

$$m(r, P(z,f)) = S(r,f).$$

Theorem 6. [A. Mohon'ko, V. Mohon'ko]
Let

$$P(z, f, f', ..., f^{(n)}) = 0$$

be an algebraic differential equation ($P(z, u_0, u_1, ..., u_n)$ is a polynomial in all arguments) and let f be its transcendental meromorphic solution. If a constant a does not solve the equation then

$$m(r, \frac{1}{f-a}) = S(r,f) \qquad \text{and} \qquad \delta(a,f) = 0.$$

Theorem 7. *Let f be a solution of a linear equation*

$$a_n(z)f^{(n)} + a_{n-1}(z)f^{(n-1)} + ... + a_0(z)f = 0$$

such that $T(r, a_\nu) = S(r, f)$ $(\nu = 0, ..., n)$. *Then for all* $a \neq 0, \infty$ *we have* $\delta(a, f) = 0$.

If we skip the condition that the coefficients are small with respect to the solution, nothing in general can be said about deficiencies.

Theorem 8. [11] *Let g be an entire function. Then there exist entire functions a_0, a_1, a_2 such that g satisfies the equation*

$$a_2(z)f'' + a_1(z)f' + a_0(z)f = 0.$$

Theorem 9. [Clunie-type result for $\beta(a,f)$]
Let f be a transcendental meromorphic solution of

$$f^n P(z,f) = Q(z,f),$$

where n is a positive integer, $P(z,f)$, $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients a_ν, b_ν, respectively, with

$$\mathcal{L}(r,\infty,a_\nu) = S(r,f), \qquad \mathcal{L}(r,\infty,b_\nu) = S(r,f).$$

If the total degree d of $Q(z,f)$ as a polynomial in f and its derivatives is $d \le n$ then

$$\mathcal{L}(r,\infty,P(z,f)) = S(r,f).$$

Theorem 10. [Mohon'ko-Mohon'ko type result for $\beta(a,f)$]
Let

$$P(z,f,f',...,f^{(n)}) = 0 \tag{3}$$

be an algebraic differential equation with polynomial coefficients and let f be its transcendental meromorphic solution. If a constant a does not solve the equation then $\mathcal{L}(r,a,f) = S(r,f)$ and $\beta(a,f) = 0$.

4. Value distribution of solutions of Painlevé equations

Painlevé equations P_2 and P_4:

$$f'' = 2f^3 + zf + \alpha, \tag{P_2}$$

$$f'' = \frac{f'^2}{2f} + \frac{3f^3}{2} + 4zf^2 + 2(z^2 - \alpha)f + \frac{\beta}{f}, \tag{P_4}$$

Properties:

- solutions of equations P_2 and P_4 are meromorphic functions in the plane [Huhuhara, Hinkkanen and Laine, Steinmetz, Shimomura]
- the order of growth is finite [Steinmetz, Shimomura]
  (1) for a solution of $P_2 : T(r, f) = O(r^3)$, solutions of order $3/2$, rational solutions
  (2) for a solution of $P_4 : T(r, f) = O(r^4)$ solutions of order 2, rational solutions
- solutions have infinitely many poles

Theorem 11. [Schubart, Schubart and Wittich, Kiessling]
Transcendental solutions of P_2 fulfill the conditions:

Case 1, $\alpha \neq 0$,

(1) $E_N(f) = \emptyset$,

(2) • $\vartheta(0, f) \leq 1/5$,
  • $\vartheta(\infty, f) = 0$
  • $\vartheta(a, f) \leq 1/4$ *for* $a \in \mathbb{C} \setminus \{0\}$.

Case 2, $\alpha = 0$,

(1) $E_N(f) \subseteq \{0\}$ *with* $\delta(0, f) \leq 1/2$;

(2) • $\vartheta(0, f) = 0$,
  • $\vartheta(\infty, f) = 0$,
  • $\vartheta(a, f) \leq 1/4$ *for* $a \in \mathbb{C} \setminus \{0\}$.

Theorem 12. [Steinmetz]
Transcendental solutions of P_4 fulfill the conditions:

Case 1, $\beta \neq 0$,
(1) $E_N(f) = \emptyset$,
(2) • $\vartheta(0, f) = 0, \ \vartheta(\infty, f) = 0$
• *for* $a \in \mathbb{C} \setminus \{0\}, \ \vartheta(a, f) \leq 1/4$;

Case 2, $\beta = 0$,
(1) $E_N(f) \subseteq \{0\}$ *with*
• $\delta(0, f) = 1$ *if* f *satisfies* $f' = \pm(f^2 + 2zf)$;
• $\delta(0, f) \leq 1/2$ *if it does not;*
(2) • $\vartheta(0, f) = \frac{1}{2}, \ \vartheta(\infty, f) = 0,$
• *for* $a \in \mathbb{C} \setminus \{0\}, \ \vartheta(a, f) \leq 1/4.$

Theorem 13. *Transcendental meromorphic solutions of* $P_2(\alpha)$ *have the following properties:*
(1) *if* $\alpha \neq 0$ *then* $E_\Pi(f) = \emptyset$;
(2) *if* $\alpha = 0$ *then* $E_\Pi(f) \subseteq \{0\}$.

Theorem 14. *Transcendental meromorphic solutions of* $P_4(\alpha, \beta)$ *have the following properties:*
(1) *if* $\beta \neq 0$ *then* $E_\Pi(f) = \emptyset$;
(2) *if* $\beta = 0$ *then* $E_\Pi(f) \subseteq \{0\}$.

Equation P_{34} is the second order equation of the form

$$f'' = \frac{(f')^2}{2f} + Bf(2f - z) - \frac{A}{2f}, \tag{4}$$

where A and B are fixed complex parameters.

It follows from the relationship with P_2 that the solutions of P_{34} are meromorphic.

Theorem 15. *For a transcendental meromorphic solution f of $P_{34}(A, B)$,*
(1) *if* $A \neq 0$ *we have* $E_N(f) = \emptyset$;
(2) *if* $A \neq 0$ *then* $E_N(f) \subseteq \{0\}$ *with* $\delta(0, f) \leq 1/2$.

Corollary 1. *The equation $P_{4,34}$ does not admit transcendental entire solutions.*

Theorem 16. *For a transcendental meromorphic solution f of $P_{34}(A, B)$,*
(1) *if* $A \neq 0$ *we have* $E_\Pi(f) = \emptyset$;
(2) *if* $A \neq 0$ *then* $E_\Pi(f) \subseteq \{0\}$.

Theorem 17. *A transcendental meromorphic solution f of $P_{34}(A, B)$ satisfies the conditions:*

(1) *all the poles of f are double and $\vartheta(\infty, f) = 1/2$;*

(2) • *if $A \neq 0$ all the zeros of f are simple and $\vartheta(0, f) = 0$,*

• *if $A = 0$ the zeros are double and $\vartheta(0, f) \geq \frac{1}{4}$;*

(3) *if $a \in \mathbb{C} \setminus \{0\}$, we have $\vartheta(a, f) \leq \frac{1}{4}$.*

References

[1] J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc. **37** (1962), p. 17-27.

[2] A.A. Gol'dberg, I.V.Ostrowskii, *Distribution of values of meromorphic functions*, Nauka, Moscow 1970. (Russian)

[3] A.A. Gol'dberg, A.E. Eremenko, M.L. Sodin, *Exceptional values in the sense of R. Nevanlinna and in the sense of V. Petrenko, I*, Teor. Funkts., Funkts. Anal. Prilosh., **47**(1987), p.41-51. Engl. transl. J. Sov. Math. **48**(1990), no. 6, p. 648-655.

[4] A.A. Gol'dberg, A.E. Eremenko, M.L. Sodin, *Exceptional values in the sense of R. Nevanlinna and in the sense of V. Petrenko, II*, Teor. Funkts., Funkts. Anal. Prilosh., **48**(1987), p.58-70. Engl. transl. J. Sov. Math. **49**(1990), no. 2, p. 891-899.

[5] N.V. Govorov, *On Paley's problem*, Funct. Anal. and Its Applications vol. 3, no. 2 (1969), p.38-43. (Russian)

[6] V. Gromak, I. Laine and S. Shimomura, *Painlevé Differential Equations in the Complex Plane*, De Gruyter Studies in Mathematics **28**, Walter de Gruyter, Berlin, 2002.

[7] W.K. Hayman, *Meromorphic functions*, Oxford, Clarendon Press, 1964.

[8] A. Hinkkanen, I. Laine, *Solutions of the first and second Painlevé equations are meromorphic*, Journal d'Analyse Math. 79(1999), 345-377.

[9] M. Hukuhara, *Meromorfeo de la solvo de* $y'' = 6y^2 + x$, unpublished lecture notes.

[10] H. Kiessling, *Zur Werteverteilung der Lösungen algebraischer Differentialgleichungen,* Ph.D. thesis, Berlin, 1996.

[11] I. Laine, *Nevanlinna theory and complex differential equations*, Berlin, New York, De Gruyter, 1993.

[12] A.Z. Mohon'ko, V.D. Mohon'ko, *Estimates of the Nevanlinna characteristics of certain classes of meromorphic functions, and their applications to differential equations*, Sibirsk. Mat. Zh. 15(1974), p. 1305-1322. (Russian)

[13] E. Mues, R. Redhefer, *On the growth of logarithmic derivatives*, J. London Math. Soc. 8, 412-425 (1974).

[14] I.I.Marchenko and A.I.Shcherba, *On the magnitudes of deviations of meromorphic functions*, Mat. Sb. (1990), vol. 181, p.3-24. (Russian) Engl. transl. in Math. USSR Sb. (1991), vol. 69, p. 1-24.

[15] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer-Verlag, Berlin, 1970.

[16] V.P.Petrenko, *The growth of meromorphic functions of finite lower order*, Izv.Akad.Nauk SSSR 33, no.2 (1969), 414–454. (Russian)

[17] V.P. Petrenko, *Growth of meromorphic functions*, Kharkov, Vyshcha Shkola, 1978, 136 pp. (Russian)

[18] E. Picard, *Sur une propriete des fonctions entieres*, C.R. Acad. Sci. Paris 88 (1879).

[19] H. Schubart, *Zur Wertverteilung der Painlevéschen Transzendenten,* Arch. Math. (Basel) 7 (1956), 284-290.

[20] H. Schubart, H. Wittich, *Über die Lösungen der ersten beiden Painlevéschen Differentialgleichungen,* Math. Z. 66 (1957), 364-370.

[21] N. Steinmetz, *Zur Wertverteilung der Lösungen der vierten Painlevéschen Differentialgleichung*, Math. Z. **181** (1982), p. 553-561.

[22] N. Steinmetz, *On Painlevé equations I, II and IV*, Journal d'Analyse Math. 82 (2000), 363-377.

[23] N. Steinmetz, *Value distribution of the Painlevé transcendents*, Israel J. Math. 128 (2002), 29-52.

[24] S. Shimomura, *The first, the second and the fourth Painlevé trancendents are of finite order,* Proc. Japan Acad. Ser. A 77 (2001) 42-45.

[25] S. Shimomura, *Growth of the first, the second and the fourth Painlevé transcendents*, Math. Proc. Cambridge Philos. Soc. 134 (2003), 259-269.

[26] S. Shimomura, *Lower estimates for the Growth of Painlevé transcendents*, Funkcial. Ekvac. 46(2003), 287-295.

[27] G. Valiron, *Sur la distribution des valeurs des fonctions meromorphes*, Acta Math. 47 (1926), p. 117-142.