Semi-classical orthogonal polynomials and the Painlevé equations

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Outline

1. Introduction

Outline

1. Introduction
2. Properties of the **second Painlevé equation**

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + \alpha$$

- Hamiltonian structure
- Bäcklund transformations and associated difference equations
- Airy solutions

Outline

1. Introduction
2. Properties of the second Painlevé equation
3. Properties of the **fourth Painlevé equation**

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-\alpha)q + \frac{\beta}{q} \qquad \mathrm{P_{IV}}$$

- Hamiltonian structure
- Parabolic cylinder function solutions

Outline

1. Introduction
2. Properties of the second Painlevé equation
3. Properties of the fourth Painlevé equation
4. Orthogonal polynomials

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1. Introduction
2. Properties of the second Painlevé equation
3. Properties of the fourth Painlevé equation
4. Orthogonal polynomials
5. Semi-classical orthogonal polynomials and the fourth Painlevé equation
   - **Semi-classical Hermite weight**

$$\omega(x;t) = |x|^{\nu} \exp(-x^2 + tx), \qquad x, t \in \mathbb{R}, \quad \nu > -1$$

   - **Generalized Freud weight**

$$\omega(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x, t \in \mathbb{R}, \quad \nu > 0$$

Outline

1. Introduction
2. Properties of the second Painlevé equation
3. Properties of the fourth Painlevé equation
4. Orthogonal polynomials
5. Semi-classical orthogonal polynomials and the fourth Painlevé equation
6. Orthogonal polynomials on complex contours

$$\omega(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right), \qquad t > 0$$

on the curve $\mathcal{C}$ from $\mathrm{e}^{2\pi\mathrm{i}/3}\infty$ to $\mathrm{e}^{-2\pi\mathrm{i}/3}\infty$.

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2. Properties of the second Painlevé equation
3. Properties of the fourth Painlevé equation
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6. Orthogonal polynomials on complex contours
7. Conclusions

Painlevé Equations

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 6q^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 2q^3 + zq + A \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{Aq^2+B}{z} + Cq^3 + \frac{D}{q} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{(q-1)^2}{z^2}\left(Aq + \frac{B}{q}\right) + \frac{Cq}{z} + \frac{Dq(q+1)}{q-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left\{A + \frac{Bz}{q^2} + \frac{C(z-1)}{(q-1)^2} + \frac{Dz(z-1)}{(q-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

with A, B, C and D arbitrary constants.

Painlevé σ-Equations

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2+4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3+2z\frac{\mathrm{d}\sigma}{\mathrm{d}z}-2\sigma=0 \qquad \mathrm{S_I}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2+4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3+2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z}-\sigma\right)=\tfrac{1}{4}\beta^2 \qquad \mathrm{S_{II}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}-\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2+4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z}-2\sigma\right)+4z\vartheta_\infty\frac{\mathrm{d}\sigma}{\mathrm{d}z}=z^2\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z}-2\sigma+2\vartheta_0\right) \qquad \mathrm{S_{III}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2-4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z}-\sigma\right)^2+4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+2\vartheta_\infty\right)=0 \qquad \mathrm{S_{IV}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2-\left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2-z\frac{\mathrm{d}\sigma}{\mathrm{d}z}+\sigma\right]^2+4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+\kappa_j\right)=0 \qquad \mathrm{S_V}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left[z(z-1)\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right]^2+\left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left\{2\sigma-(2z-1)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right\}+\kappa_1\kappa_2\kappa_3\kappa_4\right]^2=\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+\kappa_j^2\right) \qquad \mathrm{S_{VI}}$$

where β, $\vartheta_0, \vartheta_\infty$ and $\kappa_1,\ldots,\kappa_4$ are arbitrary constants.

Classical Special Functions

- **Airy, Bessel, Whittaker, Kummer, hypergeometric functions**
- Special solutions in terms of rational and elementary functions (for certain values of the parameters)
- Solutions satisfy **linear** ordinary differential equations and **linear** difference equations
- Solutions related by **linear** recurrence relations

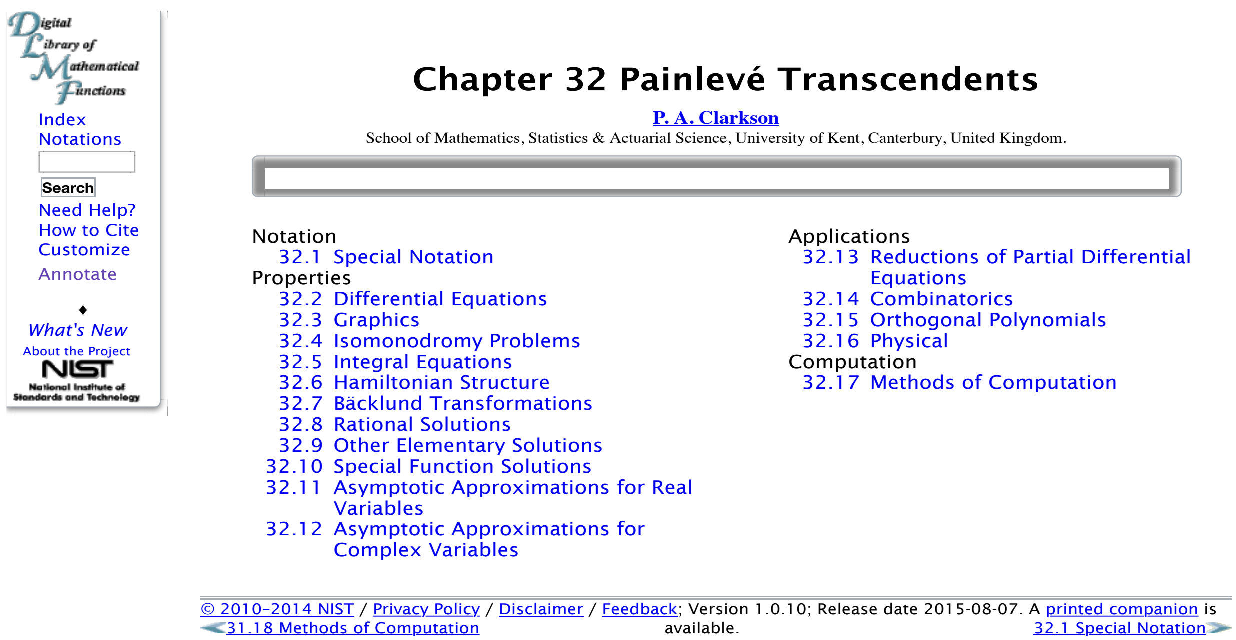
Classical Special Functions

- **Airy, Bessel, Whittaker, Kummer, hypergeometric functions**
- Special solutions in terms of rational and elementary functions (for certain values of the parameters)
- Solutions satisfy **linear** ordinary differential equations and **linear** difference equations
- Solutions related by **linear** recurrence relations

Painlevé Transcendents — Nonlinear Special Functions

- Special solutions such as rational solutions, algebraic solutions and special function solutions (for certain values of the parameters)
- Solutions satisfy **nonlinear** ordinary differential equations and **nonlinear** difference equations
- Solutions related by **nonlinear** recurrence relations

- The Painlevé equations are a chapter in the "**Digital Library of Mathematical Functions**", which is a rewrite/update of **Abramowitz & Stegun's** "**Handbook of Mathematical Functions**". This was published online, see `http://dlmf.nist.gov/32`, in 2010 and in the book "**NIST Handbook of Mathematical Functions**", by Cambridge University Press [Edited by **FWJ Olver, Lozier, Boisvert & Clark**].



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Chapter 32 Painlevé Transcendents

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31.18 Methods of Computation
32.1 Special Notation

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial
$\mathrm{P_I}$	0	—		
$\mathrm{P_{II}}$	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—
$\mathrm{P_{III}}$	2	**Bessel** $J_\nu(z), I_\nu(z), K_\nu(z)$	1	—
$\mathrm{P_{IV}}$	2	**Parabolic** $D_\nu(z)$	1	**Hermite** $H_n(z)$
$\mathrm{P_V}$	3	**Kummer** $M(a,b,z), U(a,b,z)$ **Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$
$\mathrm{P_{VI}}$	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$

Properties of the Second Painlevé Equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + \alpha \qquad \mathrm{P_{II}}$$

- **Hamiltonian structure**
- **Bäcklund transformations and associated difference equations**
- **Airy solutions**

Hamiltonian Representation

$\mathrm{P_{II}}$ can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{II}}}{\partial p} = p - q^2 - \tfrac{1}{2}z, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{II}}}{\partial q} = 2qp + \alpha + \tfrac{1}{2} \qquad \mathrm{H_{II}}$$

where $\mathcal{H}_{\mathrm{II}}(q,p,z;\alpha)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{II}}(q,p,z;\alpha) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$$

Eliminating p then q satisfies $\mathrm{P_{II}}$ whilst eliminating q yields

$$p\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{P_{34}}$$

Theorem (Okamoto [1986])

The function $\sigma(z;\alpha) = \mathcal{H}_{\mathrm{II}} \equiv \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$ *satisfies*

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

and conversely

$$q(z;\alpha) = \frac{2\sigma''(z) + \alpha + \tfrac{1}{2}}{4\sigma'(z)}, \qquad p(z;\alpha) = -2\frac{\mathrm{d}\sigma}{\mathrm{d}z}$$

is a solution of $\mathrm{H_{II}}$.

Bäcklund Transformations & Associated Difference Equations

Suppose that $q(z;\alpha)$ is a solution of $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + \alpha$$

Then the Bäcklund transformations (**Gambier [1910]**)

$$q(z;\alpha+1) = -q(z;\alpha) - \frac{2\alpha+1}{2q^2(z;\alpha)+2q'(z;\alpha)+z}$$
$$q(z;\alpha-1) = -q(z;\alpha) - \frac{2\alpha-1}{2q^2(z;\alpha)-2q'(z;\alpha)+z}$$

are also solutions of $\mathrm{P_{II}}$. Eliminating $q'(z;\alpha)$ yields

$$\frac{2\alpha+1}{q(z;\alpha+1)+q(z;\alpha)} + \frac{2\alpha-1}{q(z;\alpha)+q(z;\alpha-1)} + 4q^2(z;\alpha) + 2z = 0$$

Hence setting

$$q_{\alpha\pm1} = q(z;\alpha\pm1), \qquad q_\alpha = q(z;\alpha)$$

gives

$$\boxed{\frac{2\alpha+1}{q_{\alpha+1}+q_\alpha} + \frac{2\alpha-1}{q_\alpha+q_{\alpha-1}} + 4q_\alpha^2 + 2z = 0}$$

which is known as alt-$\mathrm{dP_I}$ (**Fokas, Grammaticos & Ramani [1993]**).

Airy Solutions of $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + \alpha \qquad \mathrm{P_{II}}$$

$$p\frac{\mathrm{d}^2 p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha+\tfrac{1}{2})^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha+\tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

Theorem

$\mathrm{P_{II}}$, $\mathrm{P_{34}}$ *and* $\mathbf{S_{II}}$ *have solutions expressible in terms of the Riccati equation*

$$\varepsilon\frac{\mathrm{d}q}{\mathrm{d}z} = q^2 + \tfrac{1}{2}z, \qquad \varepsilon = \pm 1 \tag{1}$$

if and only if $\boldsymbol{\alpha = n + \tfrac{1}{2}}$, *with* $n \in \mathbb{Z}$, *which has solution*

$$q(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\varphi(z)$$

where

$$\varphi(z) = \cos(\tfrac{1}{2}\theta)\,\mathrm{Ai}(\zeta) + \sin(\tfrac{1}{2}\theta)\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with $\mathrm{Ai}(\zeta)$ *and* $\mathrm{Bi}(\zeta)$ *the* ***Airy functions*** *and* θ *is an arbitrary constant.*

Airy Solutions of $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2q_n}{\mathrm{d}z^2} = 2q_n^3 + zq_n + n + \tfrac{1}{2} \qquad \mathrm{P_{II}}$$

$$p_n\frac{\mathrm{d}^2p_n}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2 + 2p_n^3 - zp_n^2 - \tfrac{1}{2}n^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2 \qquad \mathrm{S_{II}}$$

Theorem

Let

$$\varphi(z;\theta) = \cos(\tfrac{1}{2}\theta)\,\mathrm{Ai}(\zeta) + \sin(\tfrac{1}{2}\theta)\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with θ an arbitrary constant, $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ Airy functions, and $\tau_n(z)$ be the Wronskian

$$\tau_n(z;\theta) = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right)$$

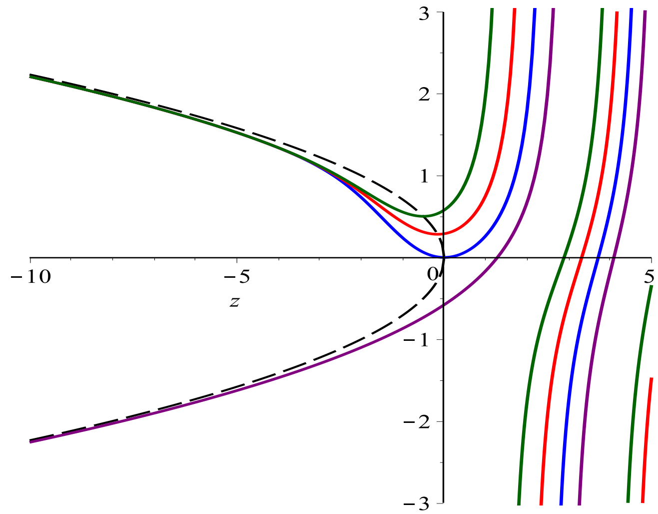
then

$$q_n(z;\theta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_n(z;\theta)}{\tau_{n+1}(z;\theta)}, \quad p_n(z;\theta) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\theta), \quad \sigma_n(z;\theta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\theta)$$

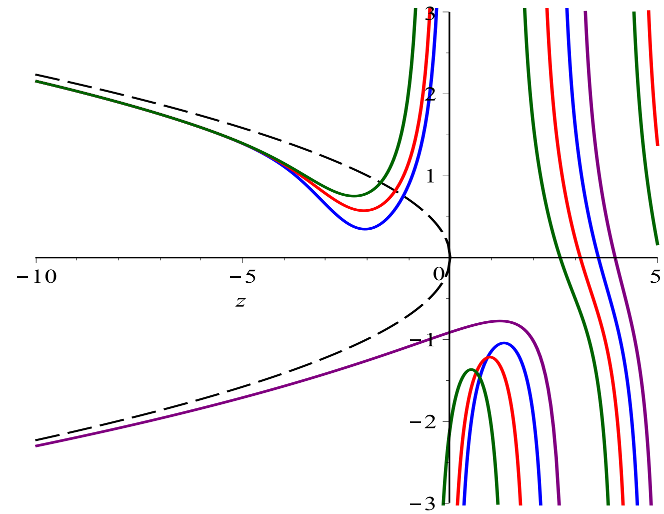
respectively satisfy $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$, with $n \in \mathbb{Z}$.

Airy Solutions of P_{II}

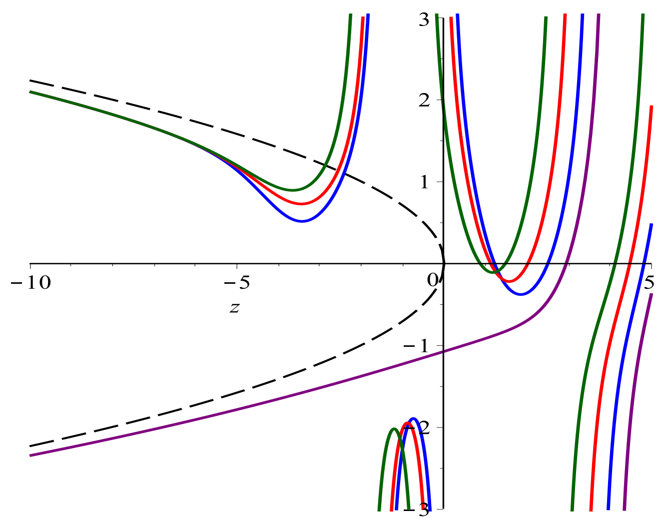
$$q_n(z;\theta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\tau_n(z;\theta)}{\tau_{n+1}(z;\theta)}$$



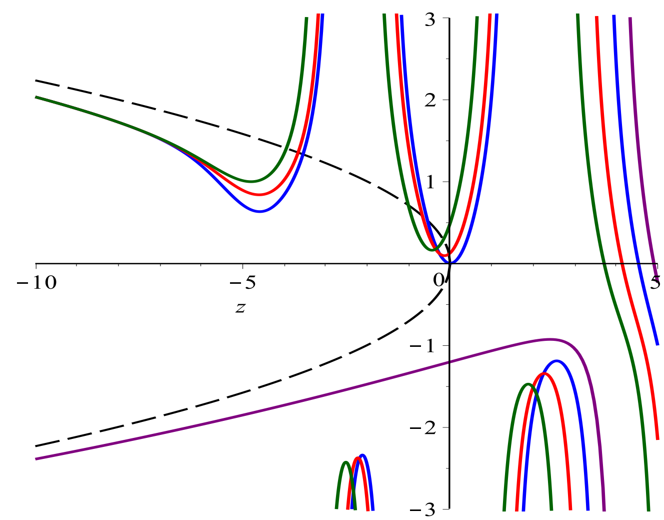
$\boldsymbol{n = 0}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n = 1}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



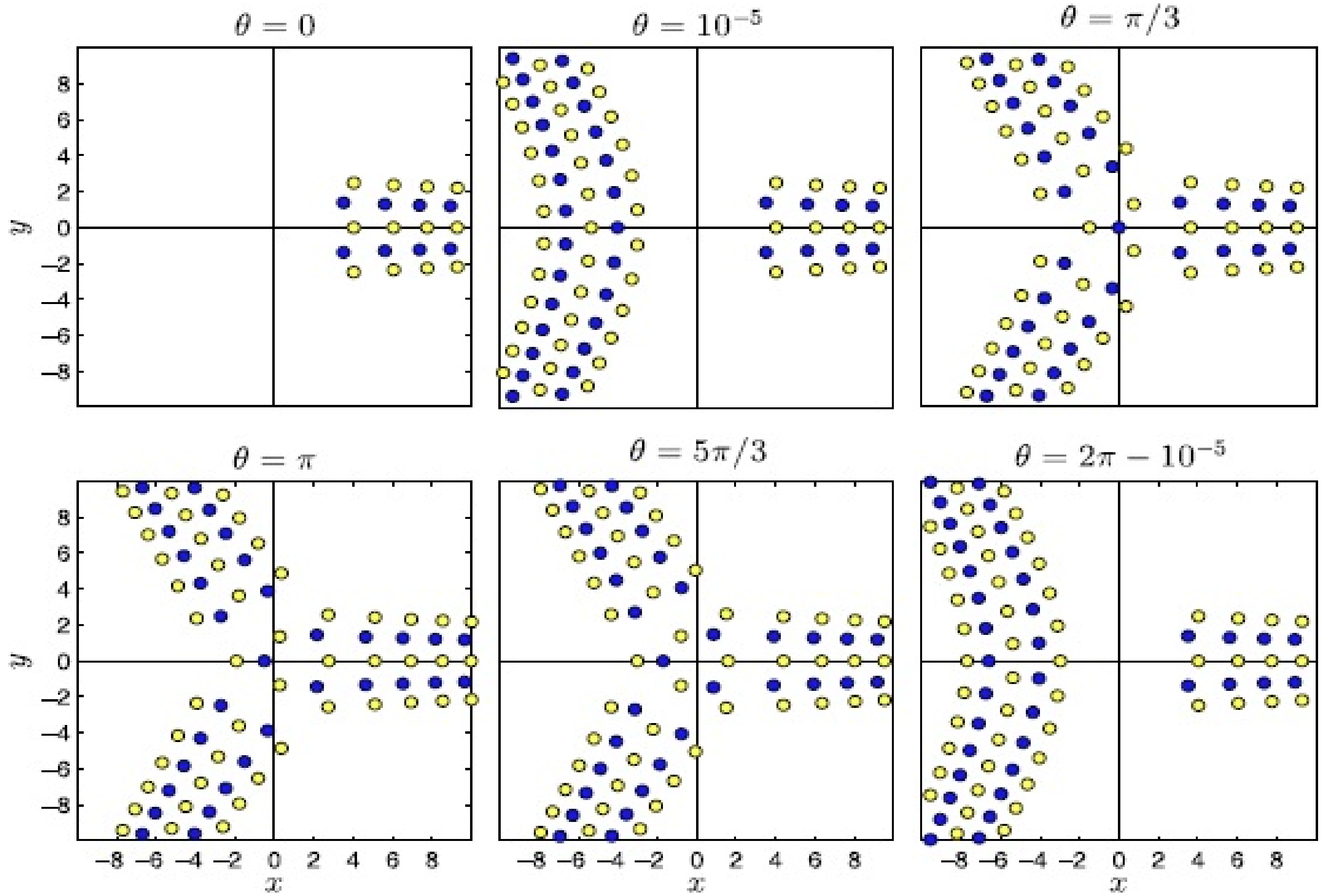
$\boldsymbol{n = 2}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n = 3}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

Airy Solutions of $\mathrm{P_{II}}$ with $\alpha = \frac{5}{2}$
(Fornberg and Weideman [2014])

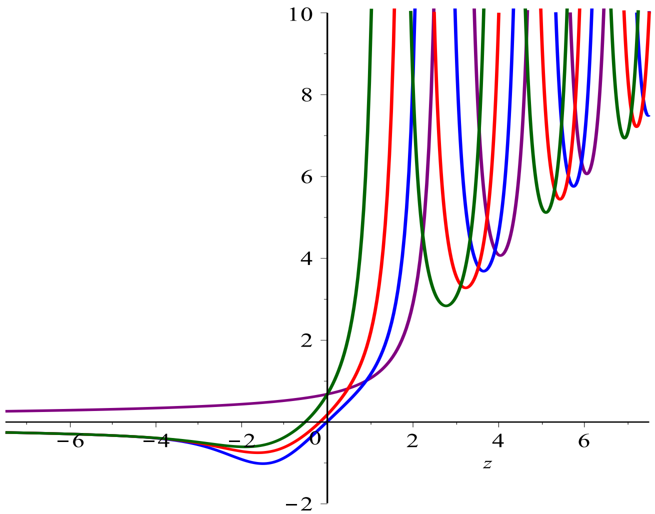
$$q(z;\tfrac{5}{2}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\mathcal{W}(\varphi,\varphi')}{\mathcal{W}(\varphi,\varphi',\varphi'')}, \qquad \varphi(z) = \cos(\tfrac{1}{2}\theta)\,\mathrm{Ai}(-2^{-1/3}z) + \sin(\tfrac{1}{2}\theta)\,\mathrm{Bi}(-2^{-1/3}z)$$



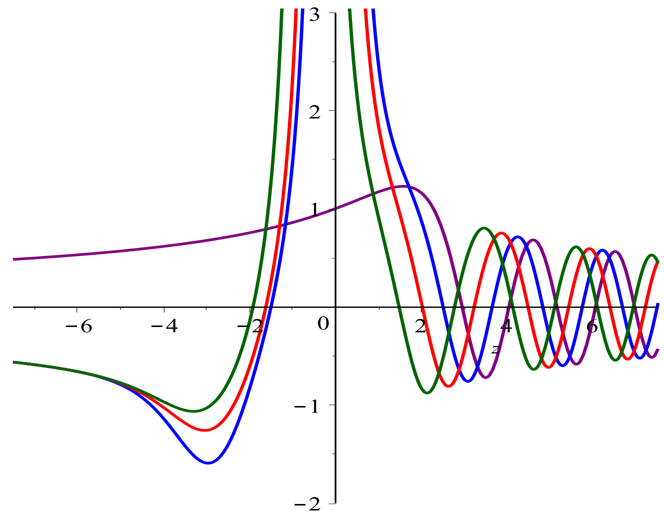
blue/**yellow** denote poles with residue $+1/-1$

Airy Solutions of P_{34}

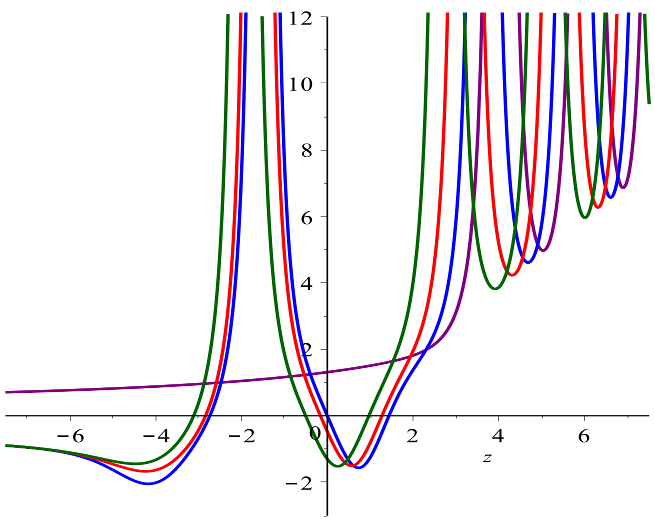
$$p_n(z;\theta) = \frac{\mathrm{d}^2}{\mathrm{d}z^2} \ln \tau_n(z;\theta)$$



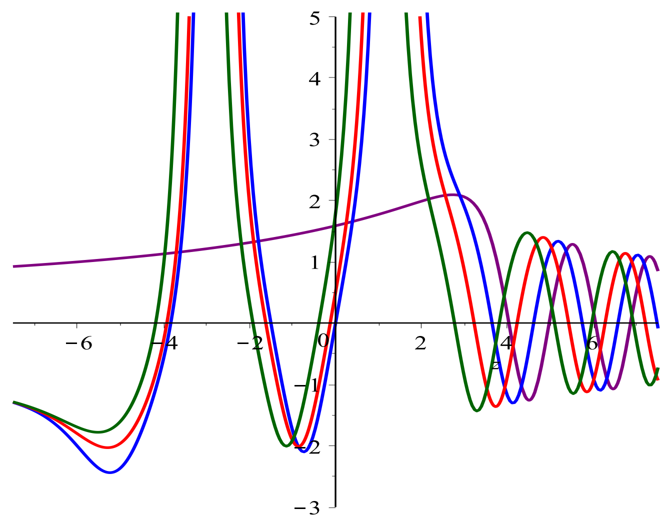
$\boldsymbol{n = 1}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n = 2}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n = 3}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n = 4}, \quad \theta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

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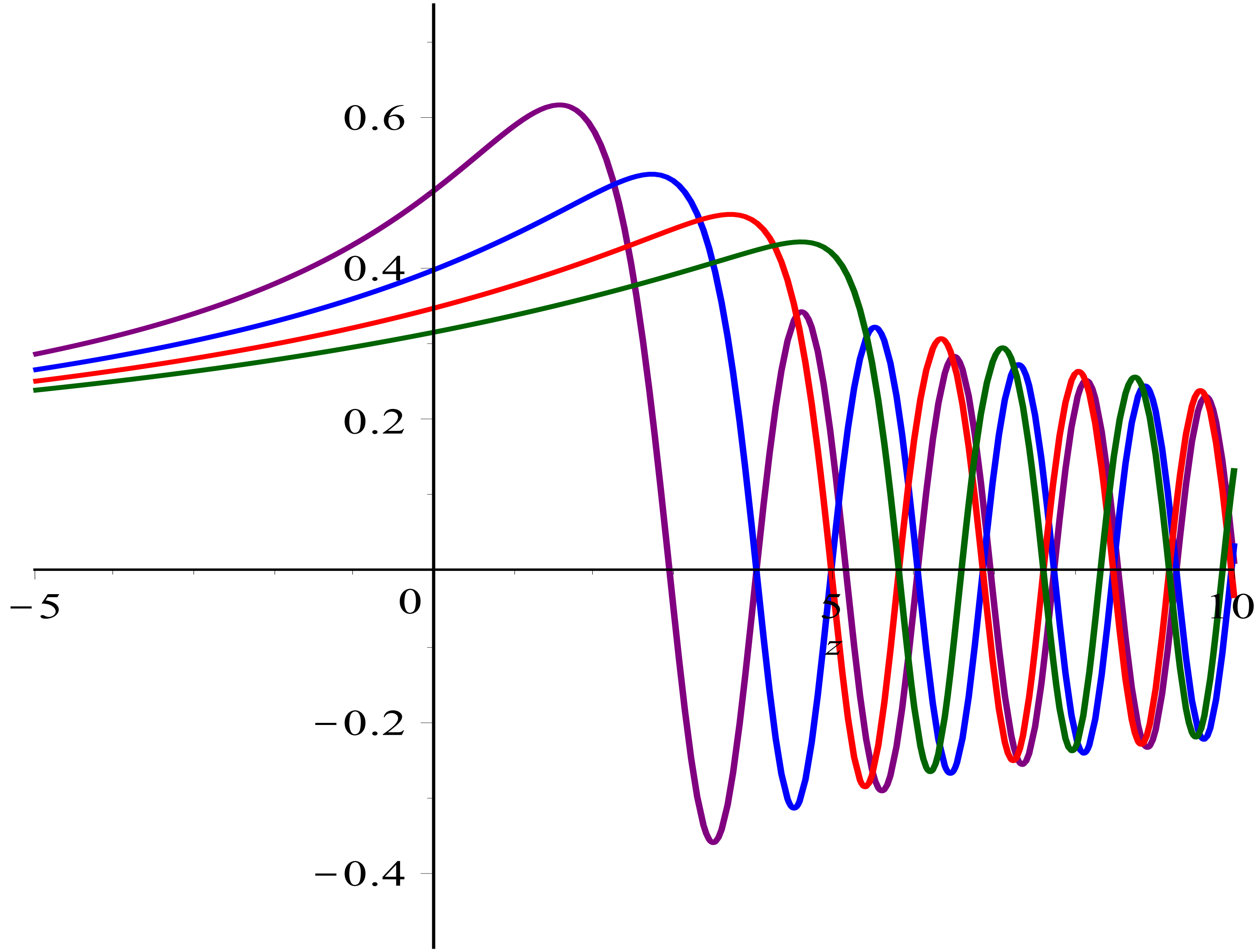
Airy Solutions of P_{34}

$$p_n(z;\theta) = \frac{\mathrm{d}^2}{\mathrm{d}z^2} \ln \tau_n(z;\theta)$$



Plots of $p(z;0)/n$ for $n = \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}$

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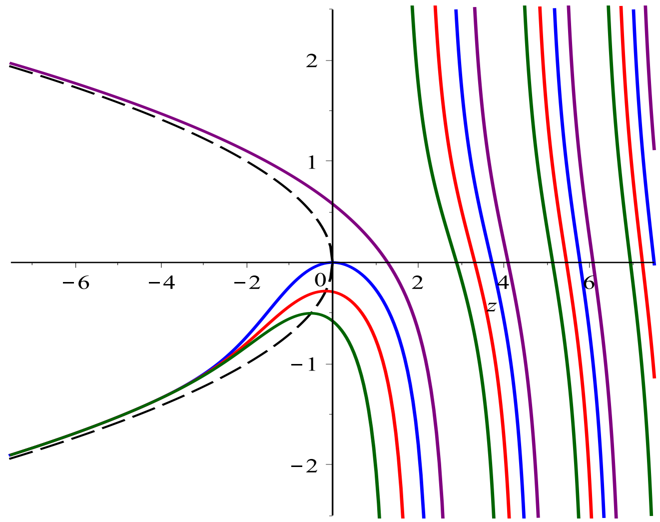
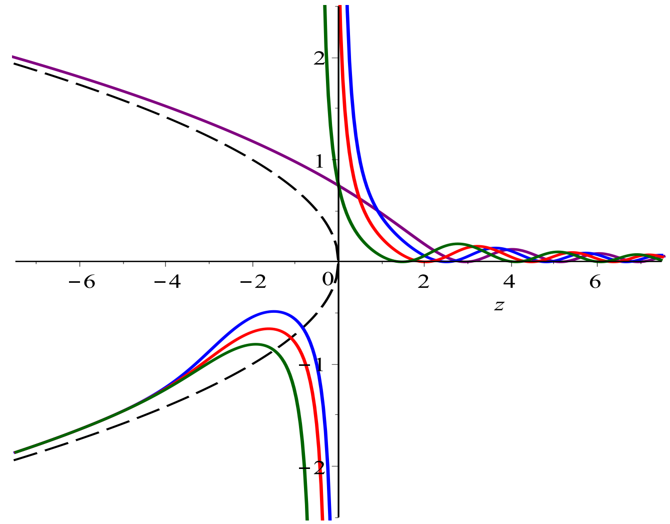
Airy Solutions of S_{II}

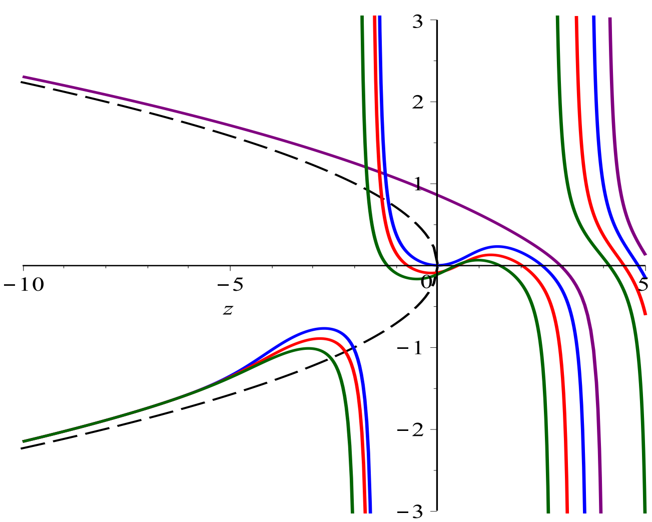
$$\sigma_n(z;\theta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \tau_n(z;\theta)$$



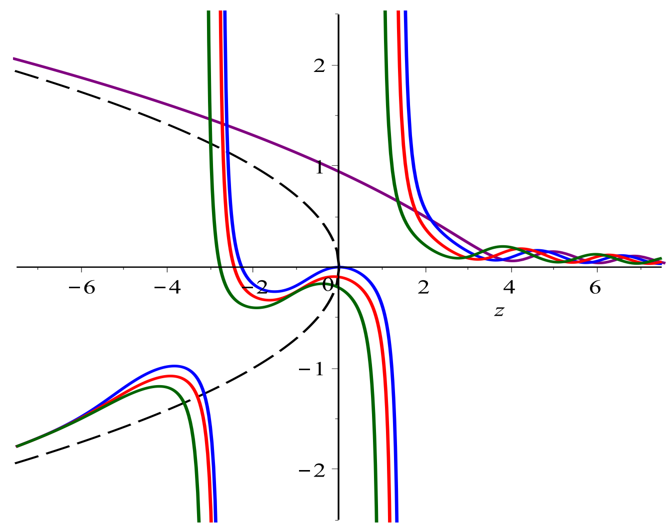
$\boldsymbol{n = 1}$, $\theta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n = 2}$, $\theta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

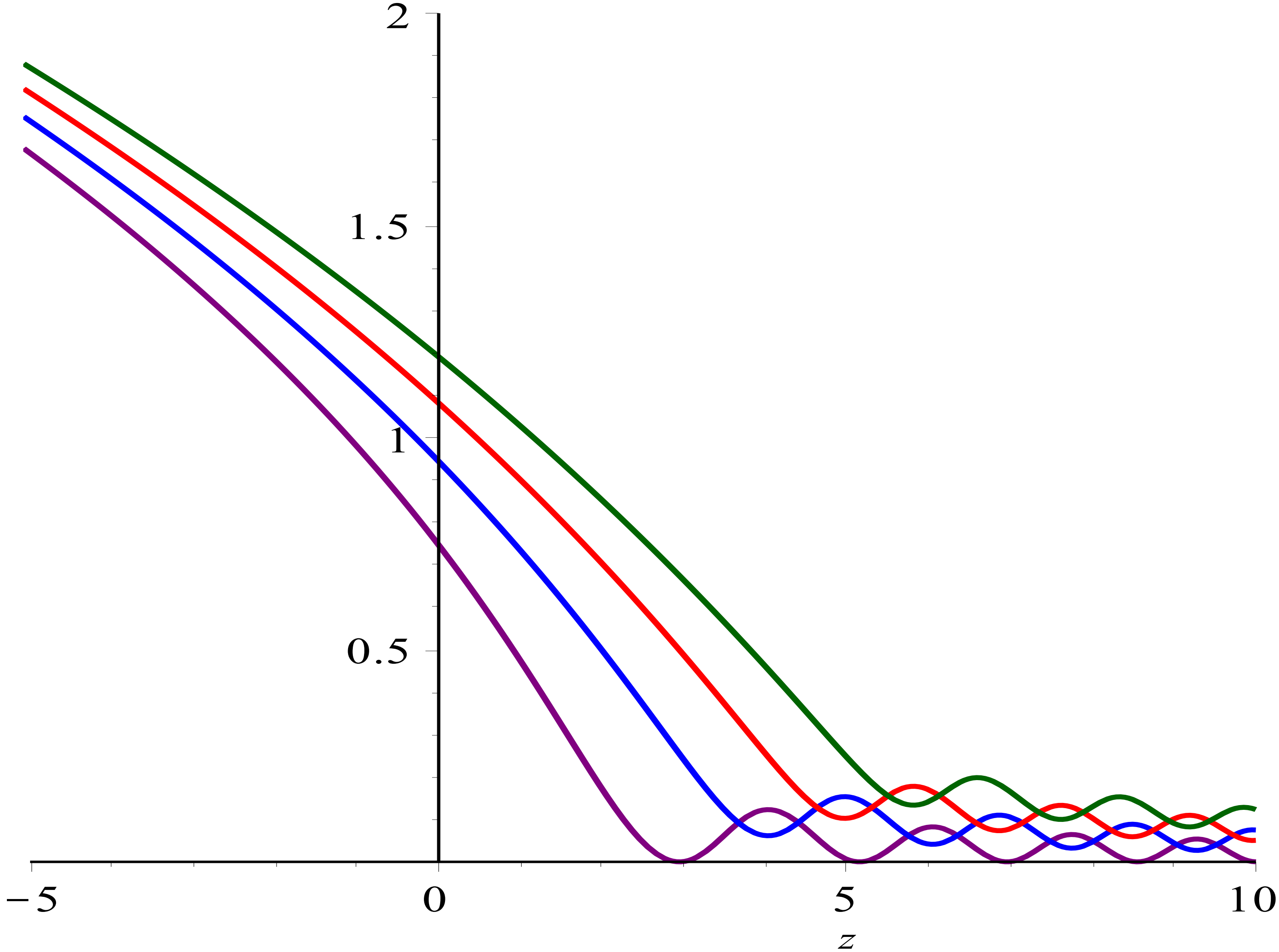


$\boldsymbol{n = 3}$, $\theta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n = 4}$, $\theta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

Airy Solutions of $\mathrm{S_{II}}$ $\sigma_n(z;0) = \dfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{W}\left(\varphi, \varphi', \ldots, \varphi^{(n-1)}\right)$, $\varphi = \mathrm{Ai}(-2^{-1/3}z)$



Plots of $\sigma_n(z;0)/n$ for $n = \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}$

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Properties of the Fourth Painlevé Equation and the Fourth Painlevé σ-Equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \qquad \mathrm{P_{IV}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

- **Hamiltonian Representation**
- **Parabolic Cylinder Function Solutions**

Hamiltonian Representation of P_{IV}

P_{IV} can be written as the **Hamiltonian system**

$$\frac{dq}{dz} = \frac{\partial \mathcal{H}_{IV}}{\partial p} = 4qp - q^2 - 2zq - 2\vartheta_0$$
$$\frac{dp}{dz} = -\frac{\partial \mathcal{H}_{IV}}{\partial q} = -2p^2 + 2pq + 2zp - \vartheta_\infty$$

where $\mathcal{H}_{IV}(q, p, z; \vartheta_0, \vartheta_\infty)$ is the Hamiltonian defined by

$$\mathcal{H}_{IV}(q, p, z; \vartheta_0, \vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

Eliminating p then q satisfies

$$\frac{d^2q}{dz^2} = \frac{1}{2q}\left(\frac{dq}{dz}\right)^2 + \tfrac{3}{2}q^3 + 4zq^2 + 2(z^2 + \vartheta_0 - 2\vartheta_\infty - 1)q - \frac{2\vartheta_0^2}{q}$$

which is P_{IV} with $A = 1 - \vartheta_0 + 2\vartheta_\infty$ and $B = -2\vartheta_0^2$, whilst eliminating q then p satisfies

$$\frac{d^2p}{dz^2} = \frac{1}{2p}\left(\frac{dq}{dz}\right)^2 + 6p^3 - 8zp^2 + 2(z^2 - 2\vartheta_0 + \vartheta_\infty + 1)p - \frac{\vartheta_\infty^2}{2p}$$

and letting $p = -\frac{1}{2}q$ gives P_{IV} with $A = 2\vartheta_0 - \vartheta_\infty - 1$ and $B = -2\vartheta_\infty^2$.

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Theorem (**Okamoto [1986]**)

The function

$$\sigma(z;\vartheta_0,\vartheta_\infty) = \mathcal{H}_{\rm IV} \equiv 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

where q *and* p *satisfy the Hamiltonian system*

$$\frac{\mathrm{d}q}{\mathrm{d}z} = 4qp - q^2 - 2zq - 2\vartheta_0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -2p^2 + 2pq + 2zp - \vartheta_\infty \qquad \mathbf{H}_{\rm IV}$$

satisfies the second-order, second-degree equation

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathbf{S}_{\rm IV}$$

Conversely, if $\sigma(z;\vartheta_0,\vartheta_\infty)$ *is a solution of* $\mathrm{S}_{\rm IV}$*, then*

$$q(z;\vartheta_0,\vartheta_\infty) = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \qquad p(z;\vartheta_0,\vartheta_\infty) = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}$$

are solutions of the Hamiltonian system $\mathrm{H}_{\rm IV}$.

Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ ***is given by***

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right), \qquad n \geq 1$$

where $\tau_{\nu,0}(z;\varepsilon) = 1$ ***and*** $\varphi_\nu(z;\varepsilon)$ ***satisfies***

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}$$

are given by

$$q_{\nu,n}^{[1]}(z) = -2z + \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \qquad (A_1, B_1) = \left(\varepsilon(2n-\nu), -2(\nu+1)^2\right)$$

$$q_{\nu,n}^{[2]}(z) = \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}, \qquad (A_2, B_2) = \left(-\varepsilon(n+\nu), -2(\nu-n+1)^2\right)$$

$$q_{\nu,n}^{[3]}(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \qquad (A_3, B_3) = \left(\varepsilon(2\nu-n+1), -2n^2\right)$$

Parabolic Cylinder Function Solutions of S_{IV}

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right), \qquad n \geq 1$$

where $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of S_{IV}

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0$$

are given by

$$\sigma_{\nu,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_{\nu,n}(z;\varepsilon), \qquad (\vartheta_0, \vartheta_\infty) = \big(\varepsilon(\nu - n + 1), -\varepsilon n\big)$$

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1 \tag{$*$}$$

- **If** $\nu \notin \mathbb{Z}$

$$\varphi_\nu(z;\varepsilon) = \begin{cases} \left\{C_1 D_\nu(\sqrt{2}\,z) + C_2 D_\nu(-\sqrt{2}\,z)\right\}\exp\left(\tfrac{1}{2}z^2\right), & \textbf{if} \quad \varepsilon = 1 \\ \left\{C_1 D_{-\nu-1}(\sqrt{2}\,z) + C_2 D_{-\nu-1}(-\sqrt{2}\,z)\right\}\exp\left(-\tfrac{1}{2}z^2\right), & \textbf{if} \quad \varepsilon = -1 \end{cases}$$

- **If** $\nu = n \in \mathbb{Z}$**, with** $n \geq 0$

$$\varphi_n(z;\varepsilon) = \begin{cases} C_1 H_n(z) + C_2 \exp(z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \textbf{if} \quad \varepsilon = 1 \\ C_1 H_n(\mathrm{i}z) + C_2 \exp(-z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \textbf{if} \quad \varepsilon = -1 \end{cases}$$

- **If** $\nu = -n \in \mathbb{Z}$**, with** $n \geq 1$

$$\varphi_{-n}(z;\varepsilon) = \begin{cases} C_1 H_{n-1}(\mathrm{i}z)\exp(z^2) + C_2 \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \textbf{if} \quad \varepsilon = 1 \\ C_1 H_{n-1}(z)\exp(-z^2) + C_2 \dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \textbf{if} \quad \varepsilon = -1 \end{cases}$$

with C_1 **and** C_2 **arbitrary constants,** $D_\nu(\zeta)$ **the parabolic cylinder function,** $H_n(z)$ **the Hermite polynomial,** $\operatorname{erfc}(z)$ **the complementary error function and** $\operatorname{erfi}(z)$ **the imaginary error function.**

Orthogonal Polynomials

- Some History
- Monic orthogonal polynomials
- Semi-classical orthogonal polynomials

Some History

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to **Shohat [1939]**, **Freud [1976]**, **Bonan & Nevai [1984]**.
- **Fokas, Its & Kitaev [1991, 1992]** identified these integrable equations as **discrete Painlevé equations**.
- **Magnus [1995]** considered the **Freud weight**

$$\omega(x;t) = \exp\left(-x^4 + tx^2\right), \qquad x, t \in \mathbb{R},$$

and showed that the coefficients in the three-term recurrence relation can be expressed in terms of solutions of

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n$$

which is **discrete** $\mathrm{P_I}$ ($\mathrm{dP_I}$), as shown by **Bonan & Nevai [1984]**, and

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = \frac{1}{2q_n}\left(\frac{\mathrm{d}q_n}{\mathrm{d}t}\right)^2 + \frac{3}{2}q_n^3 + 4tq_n^2 + 2(t^2 + \tfrac{1}{2}n)q_n - \frac{n^2}{2q_n}$$

which is $\mathrm{P_{IV}}$ with $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$. The connection between the Freud weight and solutions of $\mathrm{dP_I}$ and $\mathrm{P_{IV}}$ is due to **Kitaev [1988]**.

18.32 OP's with Respect to Freud Weights

A *Freud weight* is a weight function of the form

18.32.1
$$w(x) = \exp(-Q(x)), \quad -\infty < x < \infty,$$

where $Q(x)$ is real, even, nonnegative, and continuously differentiable. Of special interest are the cases $Q(x) = x^{2m}$, $m = 1, 2, \ldots$. No explicit expressions for the corresponding OP's are available. However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros, see Levin and Lubinsky (2001) and Nevai (1986). For a uniform asymptotic expansion in terms of Airy functions (§9.2) for the OP's in the case $Q(x) = x^4$ see Bo and Wong (1999).

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$, be the **monic orthogonal polynomials** of degree n in x, with respect to the positive weight $\omega(x)$, such that

$$\int_a^b P_m(x)P_n(x)\,\omega(x)\,\mathrm{d}x = h_n\delta_{m,n}, \quad h_n > 0, \qquad m,n = 0,1,2,\ldots$$

One of the important properties that orthogonal polynomials have is that they satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the recurrence coefficients are given by

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and $\mu_k = \displaystyle\int_a^b x^k\,\omega(x)\,\mathrm{d}x$ are the **moments** of the weight $\omega(x)$.

Further Properties

- The Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \qquad \mu_k = \int_a^b x^k \, \omega(x) \, \mathrm{d}x$$

also has the integral representation

$$\Delta_n = \frac{1}{n!} \int_a^b \cdot \overset{n}{\cdots} \cdot \int_a^b \prod_{\ell=1}^{n} \omega(x_\ell) \prod_{1 \le j < k \le n} (x_j - x_k)^2 \, \mathrm{d}x_1 \, \ldots \, \mathrm{d}x_n, \qquad n \ge 1$$

which is the **partition function** in random matrix theory.

- The monic polynomials $P_n(x)$ can be uniquely expressed as

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

- The normalization constants can be expressed as

$$h_n = \frac{\Delta_{n+1}}{\Delta_n}, \qquad h_0 = \Delta_1 = \mu_0$$

Example — Hermite polynomials

Hermite polynomials are orthogonal with respect to the weight

$$\omega(x) = \exp(-x^2), \qquad x \in \mathbb{R}$$

In this case

$$\mu_{2k} = \int_{-\infty}^{\infty} x^{2k} \exp(-x^2)\,\mathrm{d}x = \frac{\sqrt{\pi}\,(2k)!}{2^{2k}\,k!}, \qquad \mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} \exp(-x^2)\,\mathrm{d}x = 0$$

so

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = (\tfrac{1}{2})^{n(n-1)/2} \prod_{k=1}^{n-1} (k!), \qquad \widetilde{\Delta}_n = 0$$

and therefore

$$\alpha_n = 0, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = \tfrac{1}{2}n$$

which gives the three-term recurrence relation

$$P_{n+1}(x) = xP_n(x) - \tfrac{1}{2}nP_{n-1}(x)$$

where

$$P_n(x) = 2^{-n} H_n(x)$$

with $H_n(x)$ the **Hermite polynomial**.

Semi-classical Orthogonal Polynomials

Consider the **Pearson equation** satisfied by the weight $\omega(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sigma(x)\omega(x)] = \tau(x)\omega(x)$$

- **Classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Laguerre	$x^\nu \exp(-x)$	x	$1+\nu-x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$

- **Semi-classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Airy	$\exp(-\frac{1}{3}x^3 + tx)$	1	$t-x^2$
semi-classical Hermite	$\lvert x\rvert^\nu \exp(-x^2+tx)$	x	$1+\nu+tx-2x^2$
Generalized Freud	$\lvert x\rvert^{2\nu+1} \exp(-x^4+tx^2)$	x	$2\nu+2+2tx^2-4x^4$

If the weight has the form

$$\omega(x;t) = \omega_0(x)\exp(tx)$$

where the integrals $\displaystyle\int_{-\infty}^{\infty} x^k \omega_0(x)\exp(tx)\,\mathrm{d}x$ exist for all $k \geq 0$.

- The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the **Toda system**

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_n - \beta_{n+1}, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1})$$

- The kth moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k \omega_0(x)\exp(tx)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} \omega_0(x)\exp(tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k \mu_0}{\mathrm{d}t^k}$$

- Since $\mu_k(t) = \dfrac{\mathrm{d}^k \mu_0}{\mathrm{d}t^k}$, then $\Delta_n(t)$ and $\widetilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right) = \det\left[\frac{\mathrm{d}^{j+k}\mu_0}{\mathrm{d}t^{j+k}}\right]_{j,k=0}^{n-1}$$

$$\widetilde{\Delta}_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-2}\mu_0}{\mathrm{d}t^{n-2}}, \frac{\mathrm{d}^{n}\mu_0}{\mathrm{d}t^{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta_n(t)$$

Semi-classical Hermite Weight

$$\omega(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \qquad \nu > -1$$

- **PAC & K Jordaan**, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, **39** (2014) 223–254

Semi-classical Hermite weight

Consider the **semi-classical Hermite weight**

$$\omega(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \quad \nu > -1$$

- **If $\nu \notin \mathbb{N}$, then the moment $\mu_0(t;\nu)$ is given by**

$$\begin{aligned}\mu_0(t;\nu) &= \int_{-\infty}^{\infty} |x|^\nu \exp(-x^2 + tx)\,\mathrm{d}x \\ &= \int_0^\infty x^\nu \exp(-x^2 + tx)\,\mathrm{d}x + \int_0^\infty x^\nu \exp(-x^2 - tx)\,\mathrm{d}x \\ &= \frac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\Big\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)\Big\}\end{aligned}$$

since the parabolic cylinder function $D_\nu(\zeta)$ has the integral representation

$$D_\nu(\zeta) = \frac{\exp(-\frac{1}{4}\zeta^2)}{\Gamma(-\nu)}\int_0^\infty s^{-\nu-1}\exp(-\tfrac{1}{2}s^2 - \zeta s)\,\mathrm{d}s$$

- **If $\nu = 2N$, with $N \in \mathbb{N}$ then**

$$\mu_0(t;2N) = \int_{-\infty}^{\infty} x^{2N}\exp(-x^2 + tx)\,\mathrm{d}x = \sqrt{\pi}\,\big(-\tfrac{1}{2}\mathrm{i}\big)^{2N} H_{2N}\big(\tfrac{1}{2}\mathrm{i}t\big)\exp\big(\tfrac{1}{4}t^2\big)$$

since the Hermite polynomial, $H_n(z)$, has the integral representation

$$H_n(z) = \frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}(z + \mathrm{i}x)^n \exp(-x^2)\,\mathrm{d}x$$

- **If $\nu = 2N+1$, with $N \in \mathbb{N}$ then**

$$\mu_0(t; 2N+1) = \int_{-\infty}^{\infty} x^{2N}|x| \exp(-x^2 + tx)\,\mathrm{d}x = \sqrt{\pi}\,\frac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\left\{\operatorname{erf}(\tfrac{1}{2}t)\,\exp\left(\tfrac{1}{4}t^2\right)\right\}$$

as for $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{-\infty}^{\infty} & x^n|x| \exp(-x^2 + tx)\,\mathrm{d}x \\
&= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\int_0^{\infty} x\exp(-x^2 + tx)\,\mathrm{d}x + \int_0^{\infty} x\exp(-x^2 - tx)\,\mathrm{d}x\right\} \\
&= \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\left\{\int_0^{\infty} \exp(-x^2 + tx)\,\mathrm{d}x - \int_0^{\infty} \exp(-x^2 - tx)\,\mathrm{d}x\right\} \\
&= \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\left[\tfrac{1}{2}\sqrt{\pi}\ \left\{1 + \operatorname{erf}(\tfrac{1}{2}t)\right]\exp\left(\tfrac{1}{4}t^2\right) - \tfrac{1}{2}\sqrt{\pi}\ \left[1 - \operatorname{erf}(\tfrac{1}{2}t)\right]\exp\left(\tfrac{1}{4}t^2\right)\right\} \\
&= \sqrt{\pi}\,\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\left\{\operatorname{erf}(\tfrac{1}{2}t)\,\exp\left(\tfrac{1}{4}t^2\right)\right\}
\end{aligned}$$

since

$$\int_0^{\infty} \exp(-x^2 + tx)\,\mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}\ \left\{1 + \operatorname{erf}(\tfrac{1}{2}t)\right\}\exp\left(\tfrac{1}{4}t^2\right)$$

- **The moment $\mu_k(t;\nu)$ is given by**

$$\begin{aligned}\mu_k(t;\nu) &= \int_{-\infty}^{\infty} x^k |x|^\nu \exp(-x^2+tx)\,\mathrm{d}x \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} |x|^\nu \exp(-x^2+tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}\end{aligned}$$

- **The Hankel determinant $\Delta_n(t;\nu)$ is given by**

$$\Delta_n(t;\nu) = \det\Big[\mu_{j+k}(t;\nu)\Big]_{j,k=0}^{n-1} \equiv \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

where

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\Big\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)\Big\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi}\,\big(-\tfrac{1}{2}\mathrm{i}\big)^{2N} H_{2N}\big(\tfrac{1}{2}\mathrm{i}t\big)\exp\big(\tfrac{1}{4}t^2\big), & \nu = 2N \\ \sqrt{\pi}\,\dfrac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\Big\{\mathrm{erf}\big(\tfrac{1}{2}t\big)\,\exp\big(\tfrac{1}{4}t^2\big)\Big\}, & \nu = 2N+1 \end{cases}$$

Theorem (**PAC & Jordaan [2014]**)

The recurrence coefficients $\alpha_n(t;\nu)$ *and* $\beta_n(t;\nu)$ *in the three-term recurrence relation*

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t;\nu)P_n(x;t) + \beta_n(t;\nu)P_{n-1}(x;t),$$

for monic polynomials orthogonal with respect to the semi-classical Hermite weight

$$\omega(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \quad \nu > -1$$

are given by

$$\alpha_n(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\Delta_{n+1}(t;\nu)}{\Delta_n(t;\nu)}, \qquad \beta_n(t;\nu) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \Delta_n(t;\nu)$$

where $\Delta_n(t;\nu)$ *is the Hankel determinant*

$$\Delta_n(t;\nu) = \det \Big[\mu_{j+k}(t;\nu)\Big]_{j,k=0}^{n-1} \equiv \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

with

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} \Big\{ D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big) \Big\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi}\,\big(-\tfrac{1}{2}\mathrm{i}\big)^{2N} H_{2N}\big(\tfrac{1}{2}\mathrm{i}t\big) \exp\big(\tfrac{1}{4}t^2\big), & \nu = 2N \\ \sqrt{\pi}\,\dfrac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}} \left\{ \mathrm{erf}(\tfrac{1}{2}t)\, \exp\big(\tfrac{1}{4}t^2\big) \right\}, & \nu = 2N+1 \end{cases}$$

Remarks:

- The Hankel determinant $\Delta_n(t;\nu)$ satisfies the **Toda equation**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t;\nu)=\frac{\Delta_{n-1}(t;\nu)\Delta_{n+1}(t;\nu)}{\Delta_n^2(t;\nu)}$$

and the **fourth-order, bi-linear equation**

$$\Delta_n\frac{\mathrm{d}^4\Delta_n}{\mathrm{d}t^4}-4\frac{\mathrm{d}^3\Delta_n}{\mathrm{d}t^3}\frac{\mathrm{d}\Delta_n}{\mathrm{d}t}+3\left(\frac{\mathrm{d}^2\Delta_n}{\mathrm{d}t^2}\right)^2-(\tfrac{1}{4}t^2+4n+2\nu)\left\{\Delta_n\frac{\mathrm{d}^2\Delta_n}{\mathrm{d}t^2}-\left(\frac{\mathrm{d}\Delta_n}{\mathrm{d}t}\right)^2\right\}$$
$$+\tfrac{1}{4}t\Delta_n\frac{\mathrm{d}\Delta_n}{\mathrm{d}t}+\tfrac{1}{2}n(n+\nu)\Delta_n^2=0$$

- The function $S_n(t;\nu)=\dfrac{\mathrm{d}}{\mathrm{d}t}\ln\Delta_n(t;\nu)$ satisfies

$$4\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2-\left(t\frac{\mathrm{d}S_n}{\mathrm{d}t}-S_n\right)^2+4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t}-n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t}-n-\nu\right)=0$$

which is equivalent to $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation (let $S_n(t;\nu)=\frac{1}{2}\sigma(z)$, with $z=2t$), so

$$\alpha_n(t;\nu)=S_{n+1}(t;\nu)-S_n(t;\nu),\qquad \beta_n(t;\nu)=\frac{\mathrm{d}}{\mathrm{d}t}S_n(t;\nu)$$

Recurrence coefficients for $\omega(x;t) = x^2 \exp(-x^2 + tx)$

$$\alpha_0(t) = \tfrac{1}{2}t + \frac{2t}{t^2+2}$$

$$\alpha_1(t) = \tfrac{1}{2}t + \frac{4t^3}{t^4+12} - \frac{2t}{t^2+2}$$

$$\alpha_2(t) = \tfrac{1}{2}t + \frac{6t(t^4+12-4t^2)}{t^6-6t^4+36t^2+72} - \frac{4t^3}{t^4+12}$$

$$\alpha_3(t) = \tfrac{1}{2}t + \frac{8t^3(t^4+60-12t^2)}{t^8-16t^6+120t^4+720} - \frac{6t(t^4+12-4t^2)}{t^6-6t^4+36t^2+72}$$

$$\alpha_4(t) = \tfrac{1}{2}t + \frac{10t(t^8+216t^4+720-24t^6-480t^2)}{t^{10}-30t^8+360t^6-1200t^4+3600t^2+7200} - \frac{8t^3(t^4+60-12t^2)}{t^8-16t^6+120t^4+720}$$

$$\beta_1(t) = \tfrac{1}{2} - \frac{2(t^2-2)}{(t^2+2)^2}$$

$$\beta_2(t) = 1 - \frac{4t^2(t^2-6)(t^2+6)}{(t^4+12)^2}$$

$$\beta_3(t) = \tfrac{3}{2} - \frac{6(t^4-12t^2+12)(t^6+6t^4+36t^2-72)}{(t^6-6t^4+36t^2+72)^2}$$

$$\beta_4(t) = 2 - \frac{8t^2(t^4-20t^2+60)(t^8+72t^4-2160)}{(t^8-16t^6+120t^4+720)^2}$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x;t) = [x - \alpha_n(t)]P_n(x;t) - \beta_n(t)P_{n-1}(x;t), \qquad n = 0, 1, 2, \ldots$$

then

$$P_1(x;t) = x - \frac{t(t^2+6)}{2(t^2+2)}$$

$$P_2(x;t) = x^2 - \frac{t(t^4+4t^2+12)}{t^4+12}x + \frac{t^6+6t^4+36t^2-72}{4(t^4+12)}$$

$$P_3(x;t) = x^3 - \frac{3t(t^6-2t^4+20t^2+120)}{2(t^6-6t^4+36t^2+72)}x^2 + \frac{3(t^8+40t^4-240)}{4(t^6-6t^4+36t^2+72)}x - \frac{t(t^8+72t^4-2160)}{8(t^6-6t^4+36t^2+72)}$$

$$P_4(x;t) = x^4 - \frac{2t(t^8-12t^6+72t^4+240t^2+720)}{t^8-16t^6+120t^4+720}x^3 + \frac{3(t^{10}-10t^8+80t^6+1200t^2-2400)}{2(t^8-16t^6+120t^4+720)}x^2 - \frac{t(t^{10}-10t^8+120t^6-240t^4-1200t^2-7200)}{2(t^8-16t^6+120t^4+720)}x + \frac{t^{12}-12t^{10}+180t^8-480t^6-3600t^4-43200t^2+43200}{16(t^8-16t^6+120t^4+720)}$$

Generalized Freud Weight

$$\omega(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R}, \quad \nu > 0$$

- **PAC, K Jordaan, & A Kelil**, “On a generalized Freud weight", preprint (2015).

Generalized Freud weight

For the **generalized Freud weight**

$$\omega(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R}$$

the moments are

$$\begin{aligned}
\mu_0(t;\nu) &= \int_{-\infty}^{\infty} |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x \\
&= \int_0^{\infty} y^{\nu+1} \exp\left(-y^2 + ty\right) \mathrm{d}y \\
&= 2^{-(\nu+1)/2}\Gamma(\nu+1)\ \exp(\tfrac{1}{8}t^2) D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) \\
\mu_{2n}(t;\nu) &= \int_{-\infty}^{\infty} x^{2n}|x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x \\
&= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_{-\infty}^{\infty} |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x\right) \\
&= (-1)^n \frac{\mathrm{d}^n \mu_0}{\mathrm{d}t^n}, \qquad n = 1, 2, \ldots \\
\mu_{2n+1}(t;\nu) &= \int_{-\infty}^{\infty} x^{2n+1}|x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x \\
&= 0, \qquad n = 1, 2, \ldots
\end{aligned}$$

We note that

$$\begin{aligned}\mu_{2n}(t;\nu) &= \int_{-\infty}^{\infty} x^{2n}\,|x|^{2\nu+1}\exp\left(-x^4+tx^2\right)dx\\ &= \int_{-\infty}^{\infty} |x|^{2\nu+2n+1}\exp\left(-x^4+tx^2\right)dx\\ &= \mu_0(t;\nu+n).\end{aligned}$$

Also, when $\nu = n \in \mathbb{Z}^+$, then

$$D_{-n-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) = \tfrac{1}{2}\sqrt{2\pi}\,\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\left[1+\operatorname{erf}\left(\tfrac{1}{2}t\right)\right]\exp\left(\tfrac{1}{8}t^2\right)\right\},$$

where $\operatorname{erf}(z)$ is the **error function**.

Theorem (PAC, Jordaan & Kelil [2015])

The recurrence coefficient $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t),$$

is given by

$$\beta_{2n}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t;\nu+1)}{\tau_n(t;\nu)}, \qquad \beta_{2n+1}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_{n+1}(t;\nu)}{\tau_n(t;\nu+1)}$$

where $\tau_n(t;\nu)$ is the Wronskian given by

$$\tau_n(t;\nu) = \mathcal{W}\left(\phi_\nu, \frac{\mathrm{d}\phi_\nu}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\phi_\nu}{\mathrm{d}t^{n-1}}\right)$$

with

$$\phi_\nu(t) = \mu_0(t;\nu) = \frac{\Gamma(\nu+1)}{2^{(\nu+1)/2}}\exp\left(\tfrac{1}{8}t^2\right) D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)$$

Theorem (**PAC, Jordaan & Kelil [2015]**)

The recurrence coefficient $\beta_n(t)$ *in the three-term recurrence relation*

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t),$$

is given by

$$\beta_{2n}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t;\nu+1)}{\tau_n(t;\nu)}, \qquad \beta_{2n+1}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_{n+1}(t;\nu)}{\tau_n(t;\nu+1)}$$

where $\tau_n(t;\nu)$ *is the Wronskian given by*

$$\tau_n(t;\nu) = \mathcal{W}\left(\phi_\nu, \frac{\mathrm{d}\phi_\nu}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\phi_\nu}{\mathrm{d}t^{n-1}}\right)$$

with

$$\phi_\nu(t) = \mu_0(t;\nu) = \frac{\Gamma(\nu+1)}{2^{(\nu+1)/2}}\exp\left(\tfrac{1}{8}t^2\right) D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)$$

Remark: The function $S_n(t;\nu) = \dfrac{\mathrm{d}}{\mathrm{d}t}\ln\tau_n(t;\nu)$ satisfies

$$4\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2 - \left(t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n\right)^2 + 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \nu\right) = 0$$

which is equivalent to $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation, so

$$\beta_{2n}(t;\nu) = S_n(t;\nu+1) - S_n(t;\nu), \qquad \beta_{2n+1}(t;\nu) = S_n(t+1;\nu) - S_n(t;\nu+1)$$

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Theorem

The recurrence coefficients $\beta_n(t)$ *satisfy the equation*

$$\frac{\mathrm{d}^2\beta_n}{\mathrm{d}t^2} = \frac{1}{2\beta_n}\left(\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}A_n)\beta_n + \frac{B_n}{16\beta_n} \tag{1}$$

which is equivavlent to $\mathrm{P_{IV}}$, *where the parameters* A_n *and* B_n *are given by*

$$\begin{aligned} A_{2n} &= -2\nu - n - 1, \quad & A_{2n+1} &= \nu - n \\ B_{2n} &= -2n^2, & B_{2n+1} &= -2(\nu + n + 1)^2 \end{aligned}$$

Further $\beta_n(t)$ *satisfies the nonlinear difference equation*

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \tfrac{1}{2}t + \frac{2n + (2\nu+1)[1-(-1)^n]}{8\beta_n} \tag{2}$$

which is known as **discrete** $\mathrm{P_I}$.

Theorem

The recurrence coefficients $\beta_n(t)$ satisfy the equation

$$\frac{\mathrm{d}^2\beta_n}{\mathrm{d}t^2} = \frac{1}{2\beta_n}\left(\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}A_n)\beta_n + \frac{B_n}{16\beta_n} \qquad (1)$$

which is equivavlent to $\mathrm{P_{IV}}$, where the parameters A_n and B_n are given by

$$\begin{aligned} A_{2n} &= -2\nu - n - 1, \quad & A_{2n+1} &= \nu - n \\ B_{2n} &= -2n^2, & B_{2n+1} &= -2(\nu + n + 1)^2 \end{aligned}$$

Further $\beta_n(t)$ satisfies the nonlinear difference equation

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \tfrac{1}{2}t + \frac{2n + (2\nu+1)[1-(-1)^n]}{8\beta_n} \qquad (2)$$

which is known as **discrete** $\mathrm{P_I}$.

Remark: The link between the differential equation (1) and the difference equation (2) is given by the Bäcklund transformations

$$\beta_{n+1} = \frac{1}{2\beta_n}\frac{\mathrm{d}\beta_n}{\mathrm{d}t} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}, \qquad \beta_{n-1} = -\frac{1}{2\beta_n}\frac{\mathrm{d}\beta_n}{\mathrm{d}t} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}$$

with $c_n = \tfrac{1}{2}n + \tfrac{1}{4}(2\nu+1)[1-(-1)^n]$.

The first few recurrence coefficients are:

$$\beta_1(t) = \Phi_\nu$$

$$\beta_2(t) = -\frac{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1}{2\Phi_\nu}$$

$$\beta_3(t) = -\frac{\Phi_\nu}{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1} - \frac{\nu+1}{2\Phi_\nu}$$

$$\beta_4(t) = \frac{t}{2(\nu+2)} + \frac{\Phi_\nu}{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1} + \frac{(\nu+1)(t^2+2\nu+4)\Phi_\nu + (\nu+1)^2 t}{2(\nu+2)[2(\nu+2)\Phi_\nu^2 - (\nu+1)t\Phi_\nu - (\nu+1)^2]}$$

$$\beta_5(t) = -\frac{2\nu t}{\nu+1} - \frac{2(\nu+1)}{t} - \frac{2\nu(2t^2+\nu+1)\Phi_\nu - 4\nu^2 t}{(\nu+1)[(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2]} - \frac{2[\nu t^2 + (\nu+1)(2\nu+1)]\Phi_\nu^2 + 2\nu t(t^2+4\nu+5)\Phi_\nu - 4\nu^2 t^2 - 8\nu^2(\nu+1)}{t[t\Phi_\nu^3(t) + (2t^2-2\nu+1)\Phi_\nu^2 - 6\Phi_\nu \nu t + 4\nu^2]}$$

where

$$\Phi_\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\left\{D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)\exp\left(\tfrac{1}{8}t^2\right)\right\} = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\nu}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}{D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}.$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t)P_{n-1}(x;t), \qquad n = 0, 1, 2, \ldots$$

with

$$P_0(x;t) = 1, \qquad P_{-1}(x;t) = 0, \qquad \beta_0(t) = 0$$

then the first few polynomials are given by

$$\begin{aligned}
P_1(x;t) &= x \\
P_2(x;t) &= x^2 - \Phi_\nu \\
P_3(x;t) &= x^3 + 2\frac{t\Phi_\nu - \nu}{\Phi_\nu}\, x \\
P_4(x;t) &= x^4 + 2\frac{t\Phi_\nu^2 + (2t^2+1)\Phi_\nu - 2\nu t}{\Phi_\nu^2 + 2t\Phi_\nu - 2\nu}\, x^2 - 2\frac{(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2}{\Phi_\nu^2 + 2t\Phi_\nu - 2\nu} \\
P_5(x;t) &= x^5 + 2\frac{(\nu+2)t\Phi_\nu^2 + \nu(2t^2-1)\Phi_\nu - 2\nu^2 t}{(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2}\, x^3 \\
&\qquad + \frac{2\left[2t^2 - (\nu+1)^2\right]\Phi_\nu^2 - 4\nu(\nu+3)t\Phi_\nu + 4(\nu+2)\nu^2}{(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2}\, x
\end{aligned}$$

where

$$\Phi_\nu(t) = 2t - \sqrt{2}\,\frac{D_{1-\nu}\left(\sqrt{2}\,t\right)}{D_{-\nu}\left(\sqrt{2}\,t\right)}$$

Theorem (PAC, Jordaan & Kelil [2015])

Suppose that the monic polynomials $Q_n(x;t)$ *are generated by the three-term recurrence relation*

$$xQ_n(x;t) = Q_{n+1}(x;t) + \tfrac{1}{4}[1-(-1)^n]t\,Q_{n-1}(x;t),$$

with $Q_0(x;t)=1$ *and* $Q_1(x;t)=x$ *and the monic polynomials* $P_n(x;t)$ *arise from the generalized Freud weight*

$$\omega(x;t) = |x|^{2\nu+1}\exp\left(-x^4+tx^2\right)$$

Then

$$Q_{2n}(x;t) = (x^2-\tfrac{1}{2}t)^n$$
$$Q_{2n+1}(x;t) = x(x^2-\tfrac{1}{2}t)^n$$

and in the limit as $t\to\infty$

$$P_{2n}(x;t) \to (x^2-\tfrac{1}{2}t)^n = Q_{2n}(x;t)$$
$$P_{2n+1}(x;t) \to x(x^2-\tfrac{1}{2}t)^n = Q_{2n+1}(x;t)$$

This is due to the fact that for the generalized Freud weight, as $t\to\infty$

$$\beta_{2n}(t)\to 0,\qquad \beta_{2n+1}(t)\to \tfrac{1}{2}t$$

Theorem (PAC, Jordaan & Kelil [2015])

For the generalized Freud weight

$$\omega(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right)$$

the monic orthogonal polynomials $P_n(x;t)$ *satisfy the differential-difference equation*

$$\frac{\mathrm{d}P_n}{\mathrm{d}x}(x;t) = -B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t)$$

where

$$A_n(x;t) = 4\left(x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}\right)\beta_n$$

$$B_n(x;t) = 4x\beta_n + \frac{(2\nu+1)[1+(-1)^{n+1}]}{2x}$$

with $\beta_n(t)$ *the recurrence coefficient in the three-term recurrence relation*

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t)P_{n-1}(x;t)$$

Theorem (**PAC, Jordaan & Kelil [2015]**)

For the generalized Freud weight

$$\omega(x;t) = |x|^{2\nu+1}\exp\left(-x^4 + tx^2\right)$$

the monic orthogonal polynomials $P_n(x;t)$ satisfy the differential-difference equation

$$\frac{\mathrm{d}P_n}{\mathrm{d}x}(x;t) = -B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t)$$

where

$$A_n(x;t) = 4\left(x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}\right)\beta_n$$

$$B_n(x;t) = 4x\beta_n + \frac{(2\nu+1)[1+(-1)^{n+1}]}{2x}$$

with $\beta_n(t)$ the recurrence coefficient in the three-term recurrence relation

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t)P_{n-1}(x;t)$$

For Hermite polynomials $H_n(x)$ and Laguerre polynomials $L_n^{(\alpha)}(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}H_n(x) = 2nH_{n-1}(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}L_n^{(\alpha)}(x) = \frac{n}{x}L_n^{(\alpha)}x) - \frac{n+\alpha}{x}L_{n-1}^{(\alpha)}(x)$$

Theorem (**PAC, Jordaan & Kelil [2015]**)

For the generalized Freud weight

$$\omega(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right)$$

the monic orthogonal polynomials $P_n(x;t)$ *satisfy the differential equation*

$$\frac{\mathrm{d}^2 P_n}{\mathrm{d}x^2}(x;t) + R_n(x;t)\frac{\mathrm{d}P_n}{\mathrm{d}x}(x;t) + T_n(x;t)P_n(x;t) = 0$$

where

$$R_n(x;t) = -4x^3 + 2tx - \frac{2\nu+1}{x} - \frac{2x}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}}$$

$$\begin{aligned} T_n(x;t) = {} & 4nx^2 + 4\beta_n + 16(\beta_n + \beta_{n+1} - \tfrac{1}{2})(\beta_n + \beta_{n-1} - \tfrac{1}{2})\beta_n \\ & - \frac{8x^2\beta_n + (2\nu+1)[1 + (-1)^{n+1}]}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}} + (2\nu+1)[1 + (-1)^{n+1}]\left(t - \frac{1}{2x^2}\right) \end{aligned}$$

with $\beta_n(t)$ *the recurrence coefficient in the three-term recurrence relation*

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t)P_{n-1}(x;t)$$

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Theorem (**PAC, Jordaan & Kelil [2015]**)

For the generalized Freud weight

$$\omega(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right)$$

the monic orthogonal polynomials $P_n(x;t)$ *satisfy the differential equation*

$$\frac{d^2 P_n}{dx^2}(x;t) + R_n(x;t)\frac{dP_n}{dx}(x;t) + T_n(x;t)P_n(x;t) = 0$$

where

$$R_n(x;t) = -4x^3 + 2tx - \frac{2\nu+1}{x} - \frac{2x}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}}$$

$$T_n(x;t) = 4nx^2 + 4\beta_n + 16(\beta_n + \beta_{n+1} - \tfrac{1}{2})(\beta_n + \beta_{n-1} - \tfrac{1}{2})\beta_n - \frac{8x^2\beta_n + (2\nu+1)[1+(-1)^{n+1}]}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}} + (2\nu+1)[1+(-1)^{n+1}]\left(t - \frac{1}{2x^2}\right)$$

with $\beta_n(t)$ *the recurrence coefficient in the three-term recurrence relation*

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t)P_{n-1}(x;t)$$

For Hermite and Laguerre polynomials:

$$\frac{d^2 H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n = 0, \qquad \frac{d^2 L_n^{(\alpha)}}{dx^2} + \frac{\alpha+1-n}{x}\frac{dL_n^{(\alpha)}}{dx} + \frac{n}{x}L_n^{(\alpha)} = 0$$

Orthogonal polynomials on complex contours

$$\omega(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right), \qquad t > 0$$

- **PAC, A Loureiro & W Van Assche**, “Unique positive solution for the alternative discrete Painlevé I equation", arXiv:1508.04916 (2015).

Consider the semi-classical Airy weight

$$\omega(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right), \qquad t > 0$$

on the curve $\mathcal{C}$ from $\mathrm{e}^{2\pi\mathrm{i}/3}\infty$ to $\mathrm{e}^{-2\pi\mathrm{i}/3}\infty$. The moments are

$$\mu_0(t) = \int_{\mathcal{C}} \exp\left(-\tfrac{1}{3}x^3 + tx\right) \mathrm{d}x = \mathrm{Ai}(t)$$

$$\mu_k(t) = \int_{\mathcal{C}} x^k \exp\left(-\tfrac{1}{3}x^3 + tx\right) \mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathrm{Ai}(t) = \mathrm{Ai}^{(k)}(t)$$

where $\mathrm{Ai}(t)$ is the **Airy function**, the Hankel determinant is

$$\Delta_n(t) = \mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)$$

with $\Delta_0(t) = 1$, and the recursion coefficients are

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n)}(t)\big)}{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)}$$

$$\beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)$$

with

$$\alpha_0(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\mathrm{Ai}(t) = \frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \qquad \beta_0(t) = 0$$

The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the discrete system

$$\begin{aligned}(\alpha_n + \alpha_{n-1})\beta_n - n = 0\\ \alpha_n^2 + \beta_n + \beta_{n+1} - t = 0\end{aligned} \qquad (1)$$

and the differential system (Toda)

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_{n+1} - \beta_n, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1}) \qquad (2)$$

Eliminating α_{n-1} and β_{n+1} between (1) and (2) yields

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = -\alpha_n - 2\beta_n + t, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = 2\alpha_n\beta_n - n \qquad (3)$$

Letting $x_n = -\beta_n$ and $y_n = -\alpha_n$ in (1) and (2) yields

$$\begin{aligned}x_n + x_{n+1} = y_n^2 - t\\ x_n(y_n + y_{n-1}) = n\end{aligned} \qquad (4)$$

and

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = x_n(y_{n-1} - y_n), \qquad \frac{\mathrm{d}y_n}{\mathrm{d}t} = x_{n+1} - x_n \qquad (5)$$

Eliminating x_{n+1} and y_{n-1} between (4) and (5) yields

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t, \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_ny_n + n \qquad (6)$$

Consider the system

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t$$
$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_n y_n + n$$

- Eliminating x_n yields

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1$$

which is equivalent to

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + n + \tfrac{1}{2}$$

i.e. $\mathrm{P_{II}}$ with $A = n + \frac{1}{2}$.

- Eliminating y_n yields

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}$$

which is equivalent to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = \frac{1}{2v}\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - 2v^2 - zv - \frac{n^2}{2v}$$

an equation known as $\mathrm{P_{34}}$.

The Airy solutions of the equations

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1, \qquad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}, \qquad x_0(t) = 0$$

are

$$y_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t)}{\tau_{n+1}(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)}{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n)}(t)\big)}$$

$$x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\tau_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)$$

where

$$\tau_n(t) = \mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big), \qquad n \geq 1$$

and $\tau_0(t) = 1$.

As $t \to \infty$

$$y_n(t) = t^{1/2} + \frac{2n+1}{4\,t} - \frac{12n^2+12n+5}{32\,t^{5/2}} + \frac{(2n+1)(16n^2+16n+15)}{64\,t^4} + \mathcal{O}(t^{-11/2})$$

$$x_n(t) = \frac{n}{2\,t^{1/2}} - \frac{n^2}{4\,t^2} + \frac{5n(4n^2+1)}{64\,t^{7/2}} - \frac{n^2(8n^2+7)}{16\,t^5} + \mathcal{O}(t^{-13/2})$$

$$x_n + x_{n+1} = y_n^2 - t \tag{1a}$$

$$x_n(y_n + y_{n-1}) = n \tag{1b}$$

Solving (1b) for x_n and substituting in (1a) yields

$$\frac{n+1}{y_n + y_{n+1}} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t \tag{2}$$

which is known as **alt-dP$_\mathrm{I}$** (**Fokas, Grammaticos & Ramani [1993]**).

Consider the Bäcklund transformations

$$y_{n+1} = -y_n + \frac{2(n+1)}{y_n^2 + y_n' - t} \tag{3a}$$

$$y_{n-1} = -y_n + \frac{2n}{y_n^2 - y_n' - t} \tag{3b}$$

Eliminating y_n' yields alt-dP$_\mathrm{I}$ (2), whilst letting $n \to n+1$ in (3b) and substituting (3a) yields

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1 \tag{4}$$

which is equivalent to P$_\mathrm{II}$ [$y_n(t) = -2^{1/3}q(z)$, $t = -2^{-1/3}z$, with $\alpha = n + \frac{1}{2}$].

Remark: The system (1) can also be written as

$$x_{n+1} = y_n^2 - x_n - t, \qquad y_{n+1} = -y_n + \frac{n+1}{y_n^2 - x_n - t}$$

Theorem (**PAC, Loureiro & Van Assche [2015]**)

For positive values of t, there exists a unique solution of

$$x_n + x_{n+1} = y_n^2 - t$$
$$x_n(y_n + y_{n-1}) = n$$

with $x_0(t) = 0$ for which $x_{n+1}(t) > 0$ and $y_n(t) > 0$ for all $n \geq 0$. This solution corresponds to the initial value

$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}.$$

Theorem (**PAC, Loureiro & Van Assche [2015]**)

For positive values of t, there exists a unique solution of

$$x_n + x_{n+1} = y_n^2 - t$$
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$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}.$$

Theorem (**PAC, Loureiro & Van Assche [2015]**)

For positive values of t, there exists a unique solution of

$$\frac{n+1}{y_n + y_{n+1}} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t$$

for which $y_n(t) \geq 0$ for all $n \geq 0$. This solution corresponds to the initial values

$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \qquad y_1(t) = -y_0(t) + \frac{1}{y_0^2(t) - t}$$

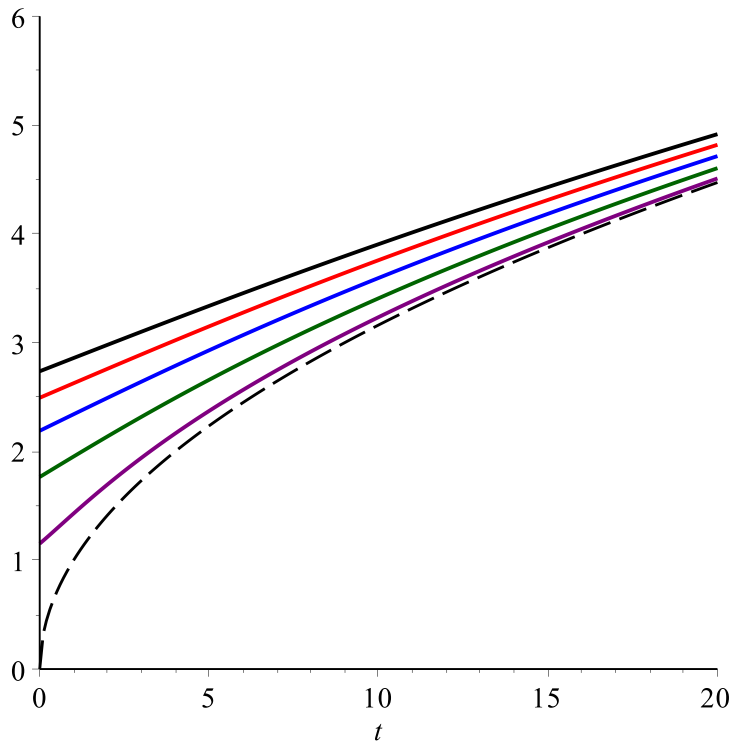
Lemma *If* $0 < t_1 < t_2$ *then*

$$y_n(t_1) < y_n(t_2), \qquad x_n(t_1) > x_n(t_2)$$

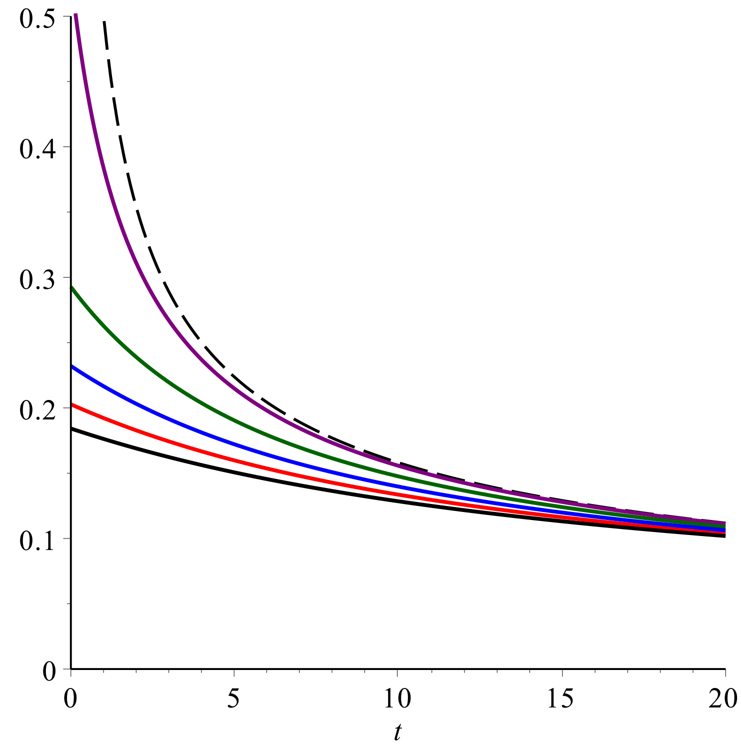
i.e. $y_n(t)$ *is monotonically increasing and* $x_n(t)$ *is monotonically decreasing.*

Lemma *For fixed* t *with* $t > 0$ *then*

$$\sqrt{t} < y_n(t) < y_{n+1}(t), \qquad \frac{1}{2\sqrt{t}} > \frac{x_n(t)}{n} > \frac{x_{n+1}(t)}{n+1}$$



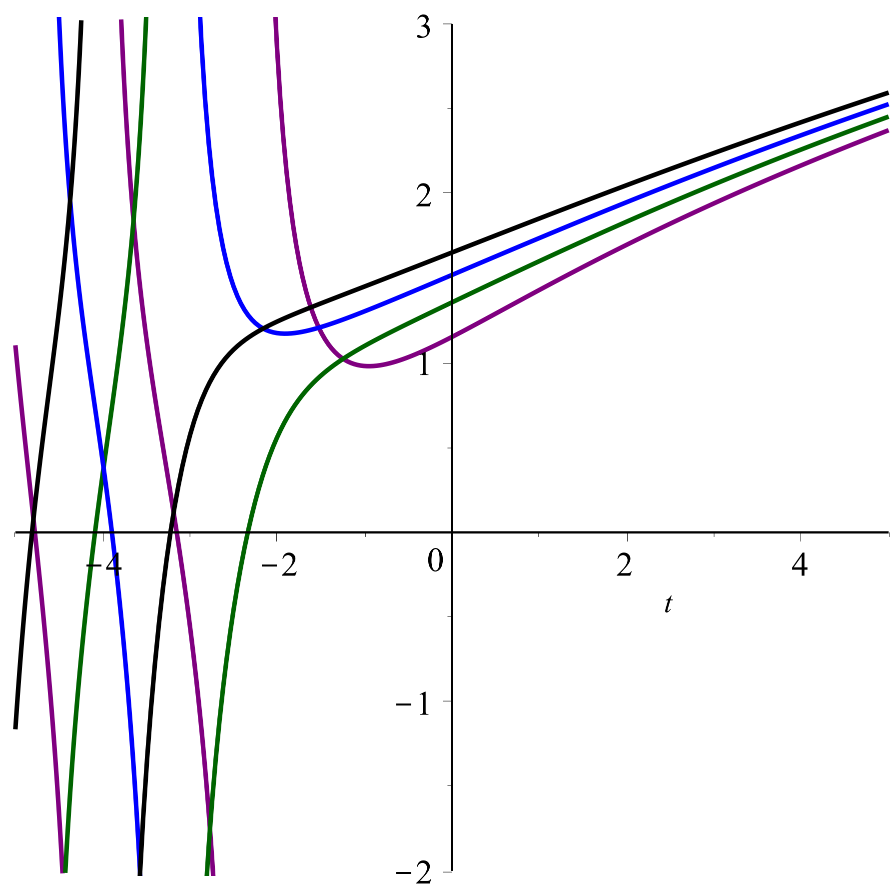
$y_n(t), \quad n = \mathbf{1}, \mathbf{5}, \mathbf{10}, \mathbf{15}, \mathbf{20}$



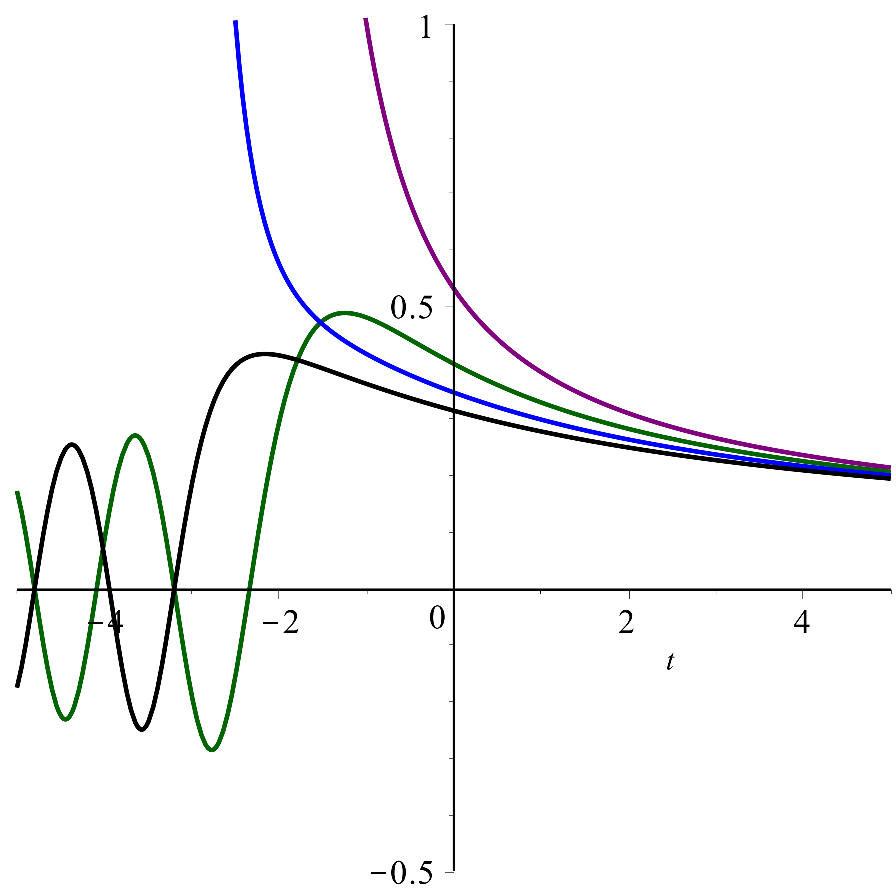
$\frac{1}{n}x_n(t), \quad n = \mathbf{1}, \mathbf{5}, \mathbf{10}, \mathbf{15}, \mathbf{20}$

Question: What happens if we don't require that $t > 0$?

$$y_n(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\tau_n(t)}{\tau_{n+1}(t)}, \qquad x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \tau_n(t), \qquad \tau_n(t) = \left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}} \operatorname{Ai}(t)\right]_{j,k=0}^{n-1}$$



$y_n(t), \quad n = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$

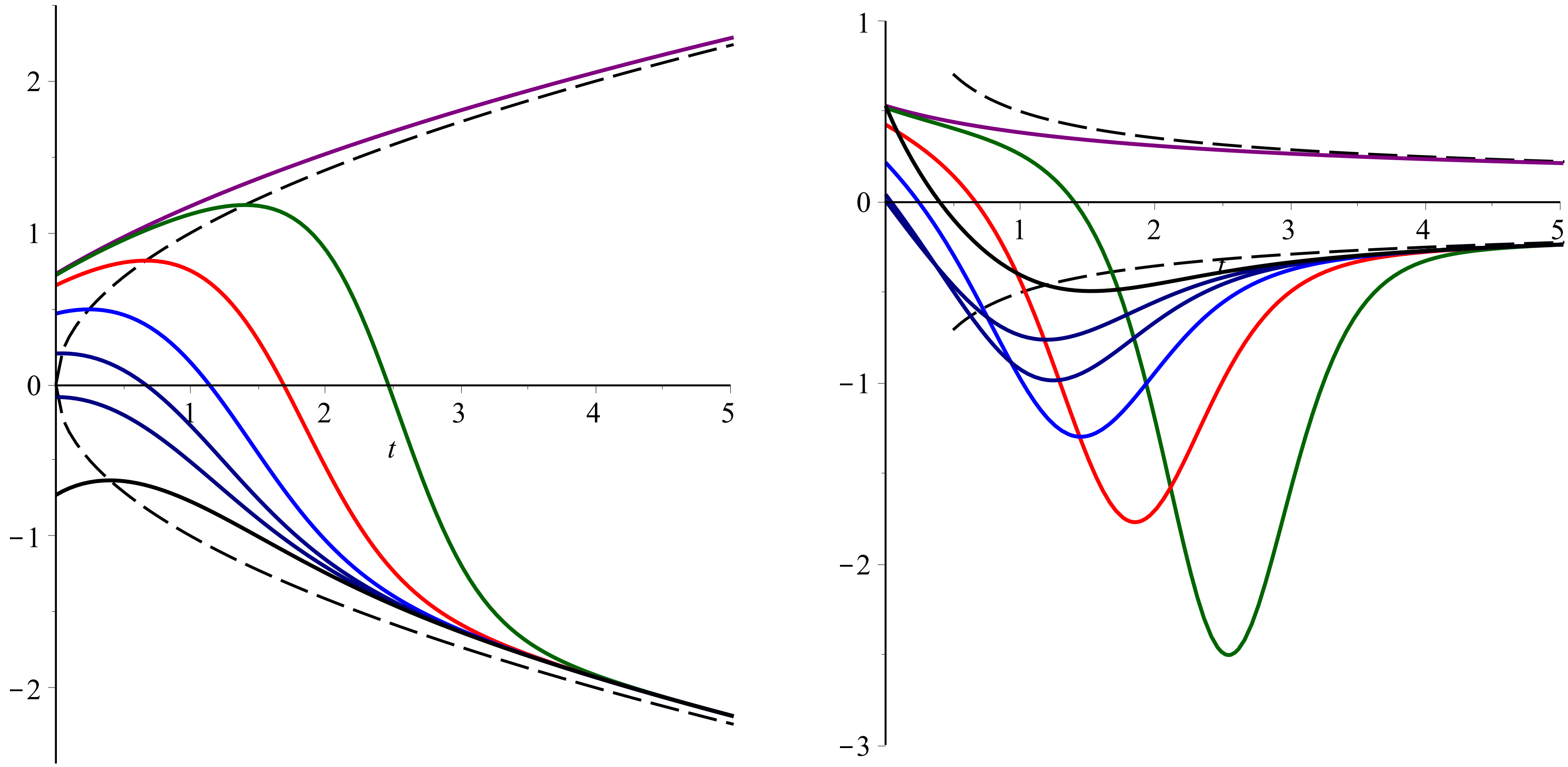


$\frac{1}{n} x_n(t), \quad n = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$

Question: What happens if we have a linear combination of $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$?

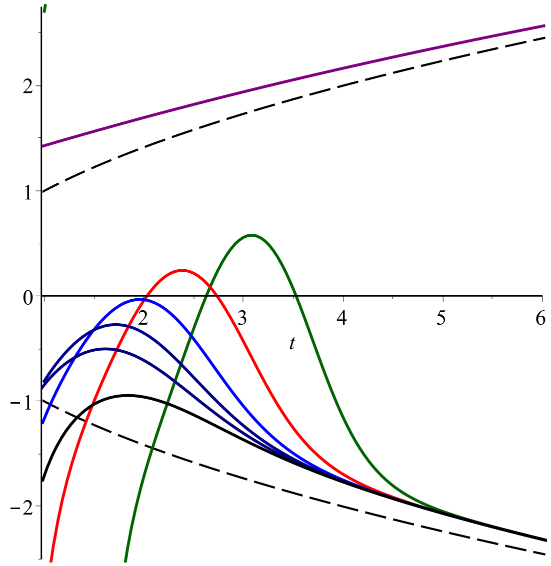
$$y_0(t;\theta) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\varphi(t;\theta), \qquad x_1(t;\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\varphi(t;\theta)$$

$$\varphi(t;\theta) = \cos(\theta)\,\mathrm{Ai}(t) + \sin(\theta)\,\mathrm{Bi}(t)$$

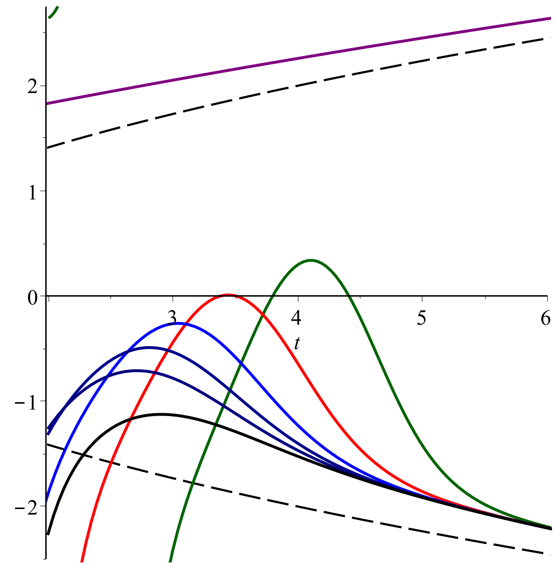


$y_0(t;\theta)$ $\qquad$ $x_1(t;\theta)$

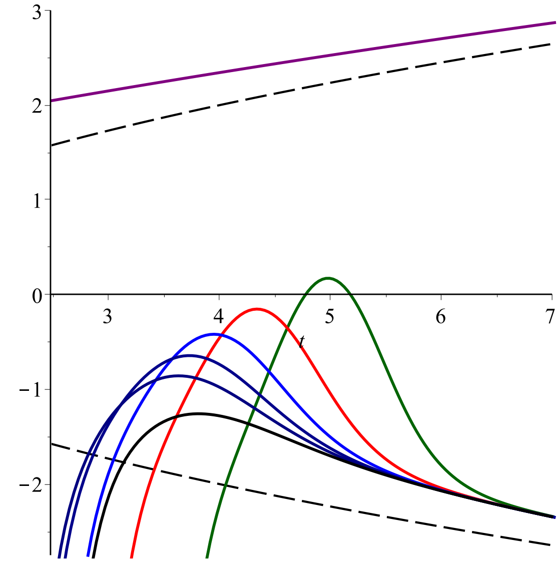
$$\theta = \mathbf{0}, \tfrac{1}{1000}\pi, \tfrac{1}{100}\pi, \tfrac{1}{25}\pi, \tfrac{1}{10}\pi, \tfrac{1}{5}\pi, \tfrac{1}{2}\pi$$



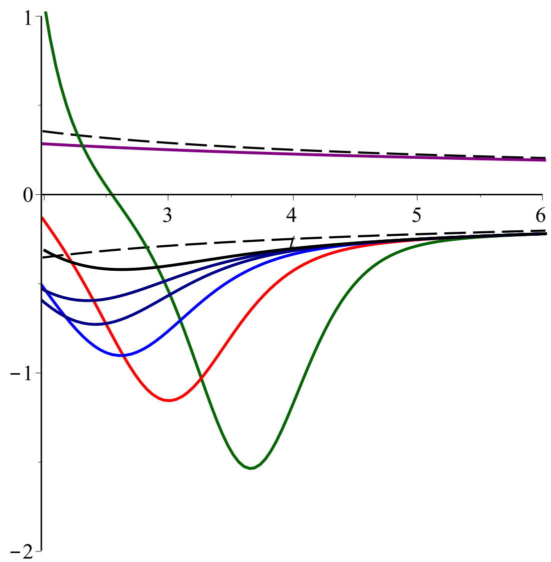
$y_1(t;\theta)$



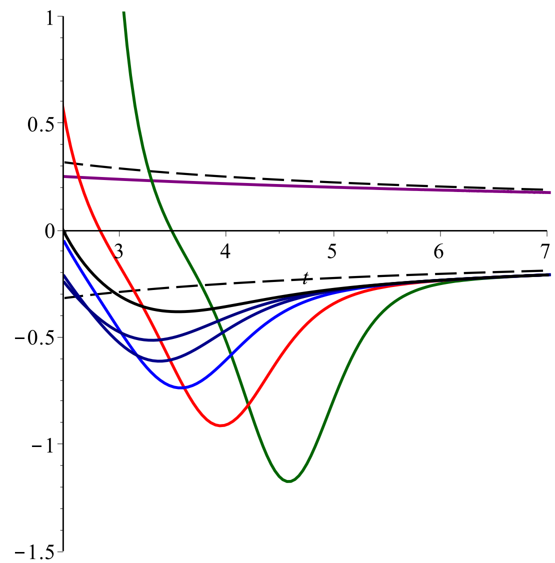
$y_2(t;\theta)$



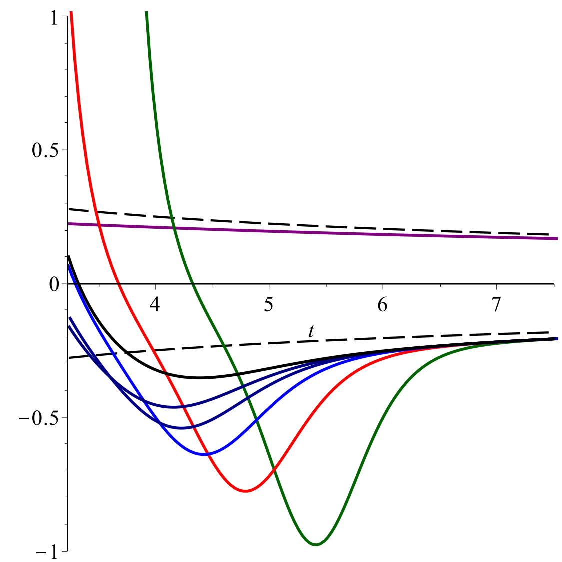
$y_3(t;\theta)$



$\frac{1}{2}x_2(t;\theta)$



$\frac{1}{3}x_3(t;\theta)$



$\frac{1}{4}x_4(t;\theta)$

Conclusions

- The coefficients in the three-term recurrence relations associated with semi-classical generalizations of orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations.
- These recursion coefficients can be expressed as Hankel determinants which arise in the solution of the Painlevé equations, in particular the Painlevé σ-equations, the second-order, second-degree equations associated with the Hamiltonian representation of the Painlevé equations.
- These Hankel determinants arise in the special cases of the Painlevé equations when they have solutions in terms of the classical special functions, the "classical solutions" of the Painlevé equations.
- The moments of the semi-classical weights provide the link between the orthogonal polynomials and the associated Painlevé equation.
- These ideas can be extended to orthogonal polynomials in other contexts:
  - **discrete orthogonal polynomials** (**PAC [2013]**); and
  - **orthogonal polynomials on the unit circle** (**PAC & Smith [2015]**).
- These results illustrate the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.

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Examples Associated with the Fifth Painlevé Equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{(q-1)^2}{z^2}\left(Aq + \frac{B}{q}\right) + \frac{Cq}{z} + \frac{Dq(q+1)}{q-1} \qquad \mathrm{P_V}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 = \left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right]^2 - 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) \qquad \mathrm{S_V}$$

The **Kummer functions** $M(a,b,z)$ and $U(a,b,z)$ have the integral representations

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \mathrm{e}^{zs}\, s^{a-1}(1-s)^{b-a-1}\,\mathrm{d}s$$

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zs}\, s^{a-1}(1+s)^{b-a-1}\,\mathrm{d}s$$

- For the **perturbed Jacobi weight** (**Basor, Chen & Ehrhardt [2010]**)

$$\omega(x;z) = x^{\alpha-1}(1+x)^{\beta-1}\mathrm{e}^{-zx}, \qquad x\in[0,1], \qquad \alpha>0,\ \ \beta>0$$

  the moments are given by

$$\mu_0(z;\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\mathrm{e}^{-z}M(\alpha,\alpha+\beta,z), \quad \mu_k(z;\alpha,\beta) = (-1)^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}\mu_0(z;\alpha,\beta)$$

- For the **Pollaczek-Jacobi weight** (**Chen & Dai [2010]**)

$$\omega(x;z) = x^{\alpha-1}(1-x)^{\beta-1}\mathrm{e}^{-z/x}, \qquad x\in[0,1], \qquad \alpha>0,\ \ \beta>0$$

  the kth moment is

$$\mu_k(z;\alpha,\beta) = \Gamma(\beta)\,\mathrm{e}^{-z}U(\beta,1-\alpha-k,z)$$

For both these weights $H_n(z) = z\dfrac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z)$, with $\Delta_n(z) = \det\Big[\mu_{j+k}(z)\Big]_{j,k=0}^{n-1}$, satisfies an equation which is equivalent to a special case of $\mathrm{S_V}$, the $\mathrm{P_V}$ σ-equation.

Theorem (**Okamoto [1987]; Forrester & Witte [2002]**)

Special function solutions of the $\mathrm{P_V}$ σ*-equation*

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad \mathrm{S_V}$$

are given by

$$\begin{aligned}\sigma(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(\varphi_{\alpha,\beta}) &- \tfrac{1}{4}(3n+2\alpha-\beta-1)z \\ &- \tfrac{5}{8}n^2 - \tfrac{1}{4}(2\alpha-3\beta-1)n - \tfrac{1}{8}(2\alpha-\beta-1)^2\end{aligned}$$

for the parameters

$$\begin{aligned}\kappa_1 &= \tfrac{1}{4}(2\alpha-\beta-n-1), & \kappa_3 &= \tfrac{1}{4}(2\alpha-\beta+3n-1)\\ \kappa_2 &= -\tfrac{1}{4}(2\alpha+\beta+n-3), & \kappa_4 &= -\tfrac{1}{4}(2\alpha-3\beta+n+1)\end{aligned}$$

where $\tau_n(\varphi_{\alpha,\beta})$ *is the determinant given by*

$$\tau_n(\varphi_{\alpha,\beta}) = \det\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j+k}\varphi_{\alpha,\beta}(z)\right]_{j,k=0}^{n-1}$$

with

$$\varphi_{\alpha,\beta}(z) = C_1M(\alpha,\beta,z) + C_2U(\alpha,\beta,z)$$

C_1 *and* C_2 *arbitrary constants,* $M(\alpha,\beta,z)$ *and* $U(\alpha,\beta,z)$ **Kummer functions**.

Deformed Laguerre weight

Consider orthogonal polynomials with the respect to the **deformed Laguerre weight**

$$\omega(x;z) = x^{\nu}(x+z)^{\lambda}\mathrm{e}^{-x}, \qquad x \in \mathbb{R}^+, \qquad \nu > 0, \quad \lambda > 0$$

Define the Hankel determinant

$$\Delta_n(z;\nu,\lambda) = \det\left[\mu_{j+k}(z;\nu,\lambda)\right]_{j,k=0}^{n-1}$$

where

$$\mu_k(z;\nu,\lambda) = \int_0^{\infty} x^{\nu+k}(x+z)^{\lambda}\mathrm{e}^{-x}\,\mathrm{d}x$$

which can be evaluated in terms of the **Kummer function** $U(a,b,z)$. **Chen & McKay [2012]** (also **Basor, Chen & McKay [2013]**) show that

$$H_n(z;\nu,\lambda) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z;\nu,\lambda)$$

satisfies

$$\left(z\frac{\mathrm{d}^2H_n}{\mathrm{d}z^2}\right)^2 = \left[(z+2n+\nu+\lambda)\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + n\lambda\right]^2 - 4\frac{\mathrm{d}H_n}{\mathrm{d}z}\left(\frac{\mathrm{d}H_n}{\mathrm{d}z}+\lambda\right)\left[z\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + n(n+\nu+\lambda)\right]$$

which is equivalent to a special case of $\mathrm{S_V}$, the $\mathrm{P_V}$ σ-equation.

Remarks

- For the **deformed Laguerre weight**

$$\omega(x;z) = x^{\nu}(x+z)^{\lambda}\mathrm{e}^{-x}, \qquad x\in\mathbb{R}^+, \qquad \nu>0,\ \ \lambda>0$$

the kth moment is

$$\begin{aligned}\mu_k(z;\nu,\lambda) &= \int_0^\infty x^{\nu+k}(x+z)^{\lambda}\mathrm{e}^{-x}\,\mathrm{d}x\\ &= z^{\nu+\lambda+k+1}\int_0^\infty s^{\nu+k}(1+s)^{\lambda}\mathrm{e}^{-sz}\,\mathrm{d}x\\ &= \Gamma(\nu+k+1)z^{\nu+\lambda+k+1}U(\nu+k+1,\nu+\lambda+k+2,t)\end{aligned}$$

with $U(a,b,z)$ the **Kummer function** of the second kind.

- In the special case of the deformed Laguerre weight when $\lambda=m\in\mathbb{Z}^+$ then

$$\begin{aligned}\mu_k(z;\nu,m) &= \int_0^\infty x^{\nu+k}(x+z)^{m}\mathrm{e}^{-x}\,\mathrm{d}x\\ &= \Gamma(\nu+k+1)z^{\nu+m+k+1}U(\nu+k+1,\nu+m+k+2,t)\\ &= \Gamma(\nu+k+1)\,(-1)^m\,m!\,L_m^{(-\nu-m-k-1)}(z)\end{aligned}$$

with $L_n^{(\alpha)}(z)$ the **Laguerre polynomial**, since

$$z^{\alpha+m}U(\alpha,\alpha+m+1,z) = (-1)^m\,m!\,L_m^{(-\alpha-m)}(z), \qquad m\in\mathbb{Z}^+$$

Discontinuous Laguerre weight

(PAC & Smith)

Consider the **discontinuous Laguerre weight**

$$\omega(x;z) = \{1-\xi\mathcal{H}(x-z)\}\,|x-z|^\lambda\,x^\nu\,\exp(-x), \qquad \nu,\lambda>0, \quad x,z\in\mathbb{R}^+$$

with $\mathcal{H}(x)$ the Heaviside step function.

Since

$$\int_0^z x^\nu(z-x)^\lambda \mathrm{e}^{-x}\,\mathrm{d}x = B(\lambda+1,\nu+1)\,z^{\nu+\lambda+1}\,\mathrm{e}^{-z}\,M(\lambda+1,\nu+\lambda+2,z)$$

$$\int_z^\infty x^\nu(x-z)^\lambda \mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(\lambda+1)\,z^{\nu+\lambda+1}\,\mathrm{e}^{-z}\,U(\lambda+1,\nu+\lambda+2,z)$$

with $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ the **Beta function**, and $M(a,b,z)$ and $U(a,b,z)$ the **Kummer functions**, then

$$\begin{aligned}\mu_0(z;\nu,\lambda) &= \int_0^\infty [1-\xi\mathcal{H}(x-z)]\,x^\nu\,|x-z|^\lambda\,\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \int_0^z x^\nu(z-x)^\lambda\,\mathrm{e}^{-x}\,\mathrm{d}x + (1-\xi)\int_z^\infty x^\nu(x-z)^\lambda\,\mathrm{e}^{-x}\,\mathrm{d}x \\ &= z^{\nu+\lambda+1}\,\mathrm{e}^{-z}\,\{B(\lambda+1,\nu+1)\,M(\lambda+1,\nu+\lambda+2,z) \\ &\qquad + (1-\xi)\Gamma(\lambda+1)U(\lambda+1,\nu+\lambda+2,z)\}\end{aligned}$$

Define the Hankel determinant

$$\Delta_n(z;\nu,\lambda) = \det\left[\mu_{j+k}(z;\nu,\lambda)\right]_{j,k=0}^{n-1}$$

then

$$H_n(z;\nu,\lambda) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z;\nu,\lambda)$$

satisfies

$$\begin{aligned} z^2\left(\frac{\mathrm{d}^2H_n}{\mathrm{d}z^2}\right)^2 &= \left[(z+2n+\nu+\lambda)\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + (2n+2\nu+\lambda)\,n\right]^2 \\ &\quad - 4\left(\frac{\mathrm{d}H_n}{\mathrm{d}z}+n\right)\left(\frac{\mathrm{d}H_n}{\mathrm{d}z}+n+\nu\right)\left[z\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + (n+\nu+\lambda)n\right] \end{aligned}$$

which is equivalent to a special case of $\mathrm{S_V}$, the $\mathrm{P_V}$ σ-equation. Specifically, letting

$$H_n(z;\nu,\lambda) = \sigma - \tfrac{1}{4}(2n+\nu-\lambda)z + \tfrac{1}{2}n^2 + \tfrac{1}{2}n(\nu+\lambda) + \tfrac{1}{8}(\nu-\lambda)^2$$

yields

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+\kappa_j\right) = 0 \qquad \mathrm{S_V}$$

with

$$\begin{aligned} \kappa_1 &= \tfrac{1}{2}n + \tfrac{3}{4}\nu + \tfrac{1}{4}\lambda, & \kappa_3 &= -\tfrac{1}{2}n - \tfrac{1}{4}\nu - \tfrac{3}{4}\lambda \\ \kappa_2 &= \tfrac{1}{2}n - \tfrac{1}{4}\nu + \tfrac{1}{4}\lambda & \kappa_4 &= -\tfrac{1}{2}n - \tfrac{1}{4}\nu + \tfrac{1}{4}\lambda \end{aligned}$$

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