Rational Solutions of the Boussinesq Equation and the Non-linear Schrödinger Equation

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Motivation

The motivation behind the investigation into the Boussinesq equations rational solutions relates to rogue waves.

To define a rogue wave consider a sea state. Take the waves which are in the top third in height and average this. A rogue wave will be twice or more than this height.

Image from "http://www.bbc.co.uk/science/horizon/2002/freakwave.shtml".

Image from "http://earthsky.org/earth/lev-kaplan-rogue-waves-are-not-tsunamis", image credit: H.Gunther and W.Rosenthal.

Rogue Waves

A rogue wave $u(x,t)$ has the following properties,

- They are localised,
- $u(x,t) \to 0$ as $(x,t) \to \infty$.

Unlike solitons they do not have the property of maintaining structure after interaction.

The Equations

The Boussinesq equation,

$$u_{tt} + u_{xx} - (u^2)_{xx} - \frac{1}{3}u_{xxxx} = 0. \tag{1}$$

The Non-Linear Schrödinger Equation,

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{2}$$

Both equations give rise to soliton solutions and rogue wave solutions.

Hirota Form

After the substitution of $u = 2(\ln F)_{xx}$ into (1) and integrating twice we obtain,

$$FF_{tt} - F_t^2 + FF_{xx} - F_x^2 - \frac{1}{3}FF_{xxxx} + \frac{4}{3}F_xF_{xxx} - F_{xx}^2 = 0. \qquad (3)$$

This can be written with Hirota's bilinear operator D as,

$$\left(D_x^2 + D_t^2 - \frac{1}{3}D_x^4\right) F \cdot F = 0. \qquad (4)$$

Polynomials F that satisfy (4) are rational solutions of (1) and thus rogue wave solutions.

The NLS solutions have a slightly different form as $u = \frac{G}{H}e^{\frac{it}{2}}$ with G and H complex polynomial solutions. The polynomial H which drives the behaviour of the waves has similarities to the F polynomial solutions to the Boussinesq equation.

The Boussinesq Equation

$$
\begin{aligned}
F_1 =& x^2 + t^2 + 1, \\
F_2 =& (x^2 + t^2 + 1)^3 + \frac{16}{3}x^4 + \left(24t^2 - \frac{152}{9}\right)x^2 + \frac{8}{3}t^4 + \frac{448}{9}t^2 + \frac{616}{9}, \\
F_3 =& (x^2 + t^2 + 1)^6 + \frac{80}{3}x^{10} + \left(200t^2 + \frac{200}{3}\right)x^8 + \left(\frac{1360}{3}t^4 + \frac{18080}{9}t^2 \right.\\
& \left. + \frac{73840}{81}\right)x^6 + \left(\frac{1280}{3}t^6 + \frac{36640}{9}t^4 + \frac{24320}{3}t^2 - \frac{5191520}{243}\right)x^4 + \\
& \left(160t^8 + \frac{34880}{9}t^6 - \frac{5440}{9}t^4 + \frac{187840}{27}t^2 + \frac{159782176}{729}\right)x^2 + \\
& \frac{40}{3}t^{10} + \frac{1400}{3}t^8 + \frac{797360}{81}t^6 + \frac{16388080}{243}t^4 + \frac{300892376}{729}t^2 \\
& + \frac{878819464}{6561}.
\end{aligned}
$$

Functions have been found for F_n up to $n = 5$ and all are of the form,

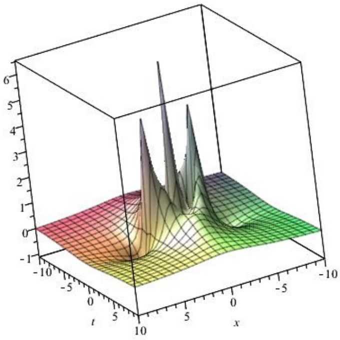
$$F_n = (x^2 + t^2 + 1)^{\frac{n(n+1)}{2}} + g(x^{\frac{n(n+1)}{2}-2}, t^{\frac{n(n+1)}{2}-2}, .., x^0, t^0), \quad (5)$$

for some function g.

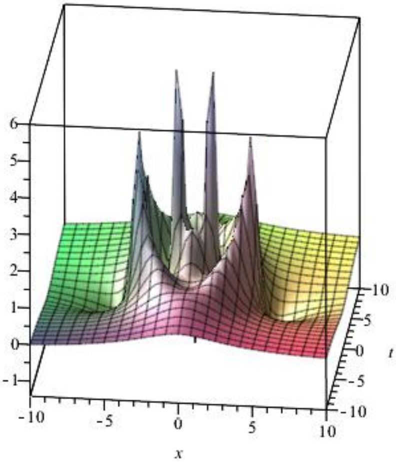
The Non-Linear Schrödinger Equation

$$
\begin{aligned}
F_1 =& x^2 + t^2 + 1,\\
F_2 =& (x^2 + t^2 + 1)^3 + \left(-24t^2 + 24\right) x^2 + 24t^4 + 96t^2 + 8,\\
F_3 =& (x^2 + t^2 + 1)^6 + \left(-120t^2 + 120\right) x^8 + \left(-240t^4 + 480t^2 + 2320\right) x^6\\
& + \left(-1440t^4 + 13440t^2 + 3360\right) x^4 + \left(240t^8 + 13440t^6 + 78240t^4\right.\\
& \left.-36480t^2 + 12144\right) x^2 + 120t^{10} + 3720t^8 + 15280t^6 + 143760t^4 +\\
& 93144t^2 + 2024.
\end{aligned}
$$

F_3 and F_4

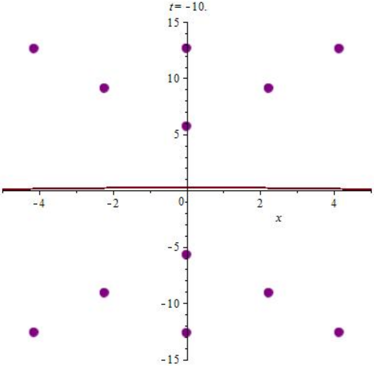


F_4 mini-waves

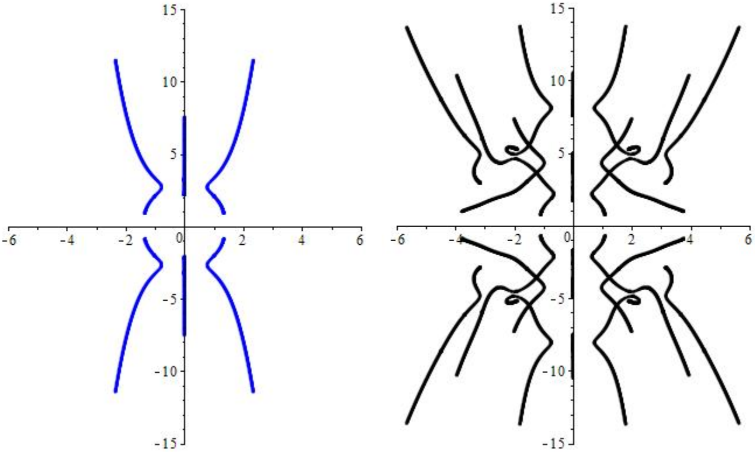


F_3

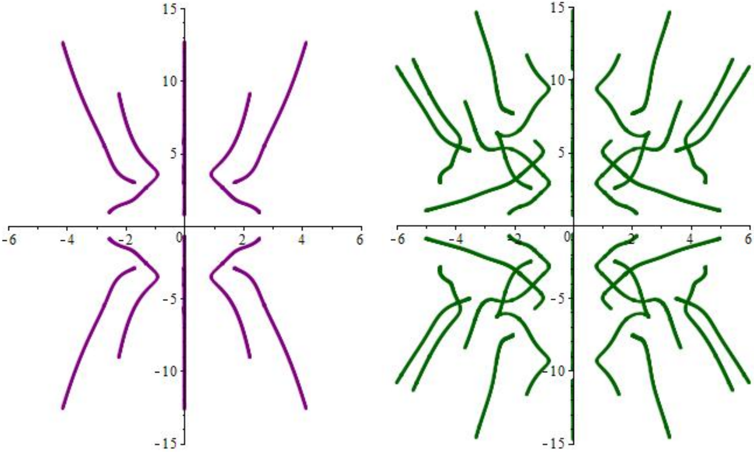
The complex roots of the rational function for the three-wave solution of the Boussinesq equation plotted on the same axes as the solution u.



Roots of F_2 and F_4



Roots of F_3 and F_5



Generalised Boussinesq Equation

We can form solutions of the Boussinesq equation in the following way,

$$\hat{F}_n = F_n + 2\alpha t P_{n-1} + 2\beta x Q_{n-1} + (\alpha^2 + \beta^2) F_{n-2}, \tag{6}$$

where α and β are real-valued parameters and P_{n-1} and Q_{n-1} are functions of x and t of degree $n(n+1)$, $P_0 = Q_0 = 0$, $F_0 = 1$.

The function,

$$tP_{n-1} \pm ixQ_{n-1}, \tag{7}$$

also satisfies the bilinear equation.

$$
\begin{aligned}
P_1 &= 3x^2 - t^2 + \frac{5}{3}, \\
Q_1 &= x^2 - 3t^2 - \frac{1}{3}, \\
P_2 &= 5x^6 + (-5t^2 + 35)x^4 + \left(-9t^4 - \frac{190}{3}t^2 - \frac{665}{9} \right) x^2 + t^6 - \frac{7}{3}t^4 \\
&\quad - \frac{245}{9}t^2 + \frac{18865}{81}, \\
Q_2 &= x^6 + \left(-9t^2 + \frac{13}{3} \right) x^4 + \left(-5t^4 - \frac{230}{3}t^2 - \frac{245}{9} \right) x^2 + 5t^6 \\
&\quad + 15t^4 + \frac{535}{9}t^2 + \frac{12005}{81}.
\end{aligned}
$$

$\hat{F}_3$ root behaviour altering α

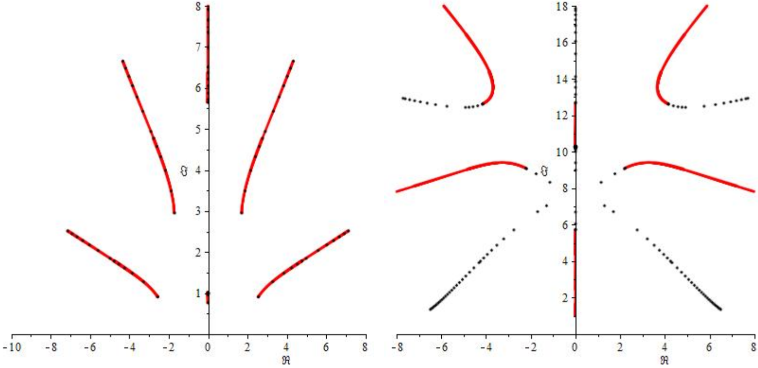


Figure: $t = 0$ and $t = 10$

For $t = 10$ values of α at turning points are:

- Between $\alpha = -1185.18$ and $\alpha = -1185.16$.
- Between $\alpha = 1938.81$ and $\alpha = 1938.83$.

$\hat{F}_3$ root behaviour altering β

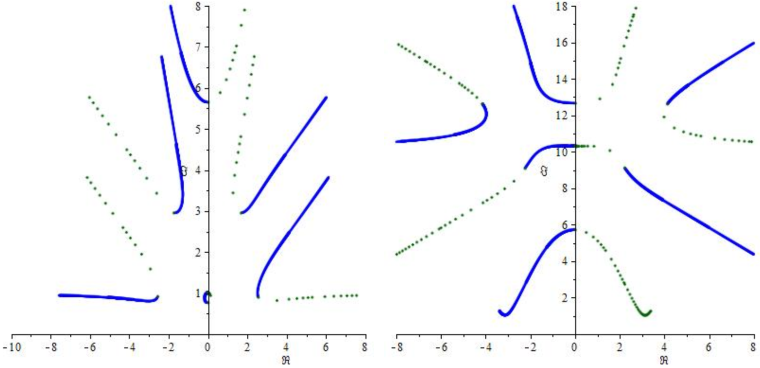
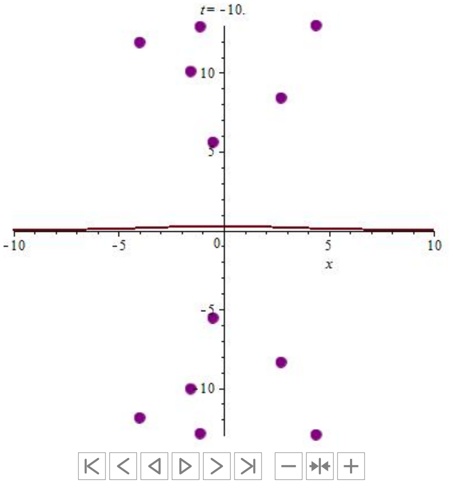


Figure: $t = 0$ and $t = 10$

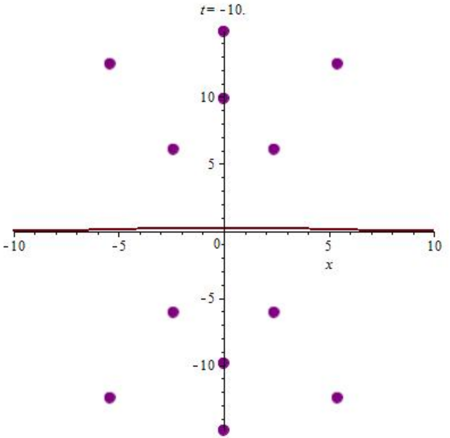
For $t = 10$ values of β at turning points are:

- Between $\beta = 834.53$ and $\beta = 834.55$ (ellipse).
- Between $\beta = -834.55$ and $\beta = -834.53$ (ellipse).
- Between $\beta = 140079.73$ and $\beta = 140079.75$.
- Between $\beta = -140079.75$ and $\beta = -140079.73$.

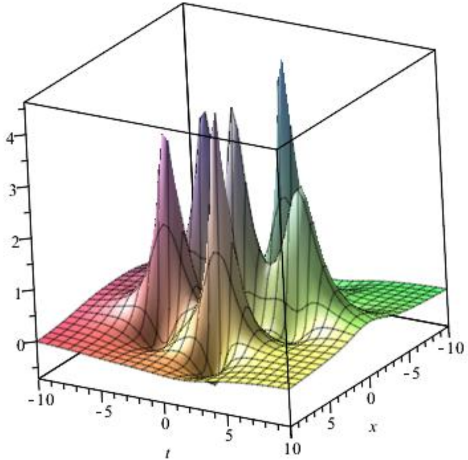
$\hat{F}_3$ with $\alpha = 0$ and $\beta = 1000$.



$\hat{F}_3$ with $\alpha = -4000$ and $\beta = 0$.



$\hat{F}_3$ 3D plot with $\alpha = -4000$ and $\beta = 1000$.



Conclusion

We have found that the rational solutions are highly structured, with the odd wave solutions seemingly embedded within each other in some way, similarly for the even wave solutions.

Targets:

- Determine a wronskian formation for the rational solutions of the Boussinesq equation.
- Establish the reason behind the structure and behaviour of the complex roots.