Computation of reduced forms of differential equations

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Abstract

- In this talk we explain how to compute the Lie algebra of the differential Galois group of some convenient $Y' = AY$, using reduced forms.
- Then, we obtain an effective way to check the Morales-Ramis-Simó criterion.

Differential Galois theory

How to compute a reduced form?

Application: effective Morales-Ramis-Simó theorem

- Let $(\mathbf{k}, \partial)$ be a field equipped with a derivation.

  $\rightarrow$ Take for example $\mathbf{k} := \mathbb{C}(z)$ with classical derivation.

- Let $C := \{\alpha \in \mathbf{k} | \partial\alpha = 0\}$ and assume that C is algebraically closed.
- We consider

$$\partial Y = AY, \text{ with } A \in \mathrm{Mat}(\mathbf{k}). \quad (1)$$

Picard-Vessiot extension

$$\partial Y = AY \text{ with } A \in \mathrm{Mat}(\mathbf{k}). \tag{1}$$

A Picard-Vessiot extension for (1) is a diff. field extension $K|\mathbf{k}$ such that

- There exists $U \in \mathrm{GL}(K)$ such that $\partial U = AU$.
- $K|\mathbf{k}$ is generated by the entries of U.
- $\{\alpha \in K | \partial\alpha = 0\} = \{\alpha \in \mathbf{k} | \partial\alpha = 0\} = C$.

Proposition

There exists an unique Picard-Vessiot extension for (1).

Differential Galois group

Definition

The differential Galois group G of (1) is the group of field automorphisms of K, commuting with the derivation and leaving all elements of $\mathbf{k}$ invariant.

$$\begin{array}{rccc} \rho_U : & G & \longrightarrow & \mathrm{GL}(C) \\ & \varphi & \longmapsto & U^{-1}\varphi(U), \end{array}$$

Theorem

The image $\rho_U(G)$ is a linear algebraic group.

Gauge transformation

Let $A \in \mathrm{Mat}(\mathbf{k})$, $P \in \mathrm{GL}(\mathbf{k})$. We have

$$\partial Y = AY \Longleftrightarrow \partial\,[PY] = P\,[A]\,PY,$$

with

$$P\,[A] := PAP^{-1} + \partial(P)P^{-1}.$$

Lie algebra of a matrix

- A *Wei-Norman decomposition* of A is a finite sum of the form
$$A = \sum a_i M_i,$$
where M_i has coefficients in C and the $a_i \in \mathbf{k}$ form a basis of the C-vector space spanned by the entries of A.
- Let $\mathrm{Lie}(A)$ be the Lie algebra generated by the M_i.

$\rightarrow$ Independent of the choice of the a_i.

- We define $\mathrm{Lie}_{\mathrm{alg}}(A) \subset \mathrm{Mat}\,(C)$ as the smallest Lie algebra of a linear algebraic group which contains $\mathrm{Lie}(A)$.

Kolchin-Kovacic reduction theorem

Theorem (Kolchin-Kovacic reduction theorem)

Assume that $\mathbf{k}$ *is a* $\mathcal{C}^1$*-field*[1] *and* G *is connected. Let* $\mathfrak{g}$ *be the Lie algebra of* G*. Let* $H \supset G$ *be a connected linear algebraic group with Lie algebra* $\mathfrak{h}$ *such that* $\mathrm{Lie}_{\mathrm{alg}}(A) \subset \mathfrak{h}$*. Then, there exists a gauge transformation* $P \in H(\mathbf{k})$ *such that* $\mathrm{Lie}_{\mathrm{alg}}(P[A]) \in \mathfrak{g}$*.*

Definition

If $\mathrm{Lie}_{\mathrm{alg}}(A) \in \mathfrak{g}$ *we will say that (1) is in reduced form.*

[1] Remind that $C(x)$ is a $\mathcal{C}^1$-field and any algebraic extension of a $\mathcal{C}^1$-field is a $\mathcal{C}^1$-field.

How to compute a reduced form?

Let us consider

$$\partial Y = \left(\begin{array}{c|c} A_1 & 0 \\ \hline A_s & A_2 \end{array}\right) Y = AY, A \in \mathrm{Mat}(\mathbf{k}). \qquad (2)$$

Assume that $\partial Y = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) Y = A_{\mathrm{diag}} Y$ is in reduced form with an abelian Lie algebra. We want to put (2) in reduced form.

Shape of the gauge transformation

Let $A_{\mathrm{sub}} := \left(\begin{array}{c|c} 0 & 0 \\ \hline A_s & 0 \end{array}\right)$.

Proposition (A-M,D,W)

There exists a gauge transformation

$$P \in \left\{ \mathrm{Id} + B, B \in \mathrm{Lie}_{\mathrm{alg}}\left(A_{\mathrm{sub}}\right) \otimes \mathbf{k} \right\},$$

such that $\partial Y = P[A]Y$ is in reduced form.

Corollary

Let $P \in \left\{ \mathrm{Id} + B, B \in \mathrm{Lie}_{\mathrm{alg}}\left(A_{\mathrm{sub}}\right) \otimes \mathbf{k} \right\}$, and assume that for all $Q \in \left\{ \mathrm{Id} + B, B \in \mathrm{Lie}_{\mathrm{alg}}\left(A_{\mathrm{sub}}\right) \otimes \mathbf{k} \right\}$, $\mathrm{Lie}(Q[P[A]]) = \mathrm{Lie}(P[A])$. Then, $\partial Y = P[A]Y$ is in reduced form.

The adjoin action

Proposition (A-M,D,W)

If $P := \mathrm{Id} + \sum f_i B_i$, *with* $f_i \in \mathbf{k}$, $B_i \in \mathrm{Lie}_{\mathrm{alg}}(A_{\mathrm{sub}})$. *Then*

$$P[A] = A + \sum f_i [B_i, A_{\mathrm{diag}}] - \sum \partial(f_i) B_i.$$

Remark

The fact that $\partial Y = A_{\mathrm{diag}} Y$ *has an abelian Lie algebra implies that we may easily compute a Jordan normal form of* $\Psi : X \mapsto [X, A_{\mathrm{diag}}]$. *Furthermore the eigenvalues of* Ψ *belongs to* $\mathbf{k}$.

Let λ_j be the eigenvalues of Ψ. We have the decomposition:

$$\mathrm{Lie}_{\mathrm{alg}}\left(A_{\mathrm{sub}}\right) = \bigoplus_{i,j} E_{\lambda_j}^{(i)} \bigcap \mathrm{Lie}_{\mathrm{alg}}\left(A_{\mathrm{sub}}\right),$$

where

$$E_{\lambda_j}^{(i)} := \ker\left(\left(\Psi - \lambda_j \mathrm{Id}\right)^i\right) / \ker\left(\left(\Psi - \lambda_j \mathrm{Id}\right)^{i-1}\right).$$

We are going to perform the reduction on the $E_{\lambda_j}^{(i)}$ separately.

Reduction with one eigenvalue λ

- Fix $m \in \mathbb{N}$. Write $A_{\text{sub}} = \bar{A} + \sum_i b_i B_i$, where $b_i \in \mathbf{k}$, B_i form a basis of $E_\lambda^{(m)} \bigcap \text{Lie}_{\text{alg}}(A_{\text{sub}})$.
- Compute a basis $\left((g_j, \underline{c}_{(\bullet,j)})\right)$ of elements in $\mathbf{k} \times C$ such that $\partial y = \lambda y + \sum_i c_{i,j} b_i$ has a solution $y = g_j \in \mathbf{k}$.
- Construct a constant invertible matrix $\overline{Q} \in \text{GL}(C)$ whose first columns are the $\underline{c}_{(\bullet,j)}$. Let $(\gamma_{i,j}) = \overline{Q}^{-1}$.
- Let $f_i := \sum_j \gamma_{i,j} g_j$. Perform $P_\lambda^{(m)} := \text{Id} + \sum_i f_i B_i$.

Reduction in general

Theorem (A-M,D,W)
Let $P := \prod_{i,j} P^{(i)}_{\lambda_j}$*. Then,* $\partial Y = P[A]Y$ *is in reduced form.*

General principle of the Morales-Ramis-Simó theorem

Hamiltonian complex system

$\downarrow$

Variational equations

$\downarrow$

Differential Galois groups

Linearization

General principle of the Morales-Ramis-Simó theorem

Integrable Hamiltonian complex system

$\downarrow$ Linearization

Variational equations

$\downarrow$

Differential Galois groups with abelian Lie algebra

Theorem (Morales-Ramis-Simó)

Let us consider an Hamiltonian system and let G_p be the differential Galois group of the variational equation of order p. If the Hamiltonian system is integrable, then for all p, the Lie algebra of G_p is abelian.

Shape of the variational equations

Let $\partial Y = A_p Y$ be the variational equation of order p. We have

$$A_p := \left(\begin{array}{c|c} \mathrm{sym}^p(A_1) & 0 \\ \hline S_p & A_{p-1} \end{array} \right) \in \mathrm{Mat}(\mathbb{C}(x)).$$

Reduction of $\partial Y = A_{p+1} Y$

- Let $p \in \mathbb{N}$. Assume that $\partial Y = A_p Y$ is in reduced form and G_p has an abelian Lie algebra.
- We use our previous work to put $\partial Y = A_{p+1} Y$ in reduced form.
- If G_{p+1} has an abelian Lie algebra, we may put $\partial Y = A_{p+2} Y$ in reduced form.