Special values of hypergeometric series

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Intro 0

Our Aim:
"to systematically find hypergeometric series(HGS) which can be expressed in terms of well-known functions".

But, before aiming for our goal,
we should answer the question "Why do we find such series?".
For this, we start from observation of the binomial theorem.

Intro 1:the binomial theorem

Denote the Pochhammer symbol by

$$(a)_i := \frac{\Gamma(a+i)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+i-1) & \text{if } i = 1, 2, 3, \cdots \\ 1 & \text{if } i = 0. \end{cases}$$

Then, the binomial theorem is following:

$$\begin{aligned} {}_1F_0(a; -; x) &:= \sum_{i=0}^{\infty} \frac{(a)_i}{i!} x^i = \sum_{i=0}^{\infty} \binom{-a}{i} (-x)^i \\ &= (1-x)^{-a} \quad (|x| < 1). \end{aligned}$$

Remark 1:

- Appropriate series (in this instance, ${}_1F_0$) can be expressed in terms of well-known functions. That is, we can get the value!

Intro 2: application of the binomial theorem 1

The binomial theorem has many application.
By substituting $a = 1/2$, $x = 1/50$ into the binomial thereom,

$$ {}_1F_0\left(\frac{1}{2}; -; \frac{1}{50}\right) = \left(\frac{49}{50}\right)^{-\frac{1}{2}} = \frac{5\sqrt{2}}{7}. $$

Namely,

$$ \sqrt{2} = \frac{7}{5}\left\{1 + \frac{1}{100} + \frac{1\cdot 3}{2!}\left(\frac{1}{100}\right)^2 + \frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{100}\right)^3 + \cdots\right\}. $$

From this, we can perform calculation of $\sqrt{2}$.
Remark 2:

- We can do numerical calculation of algebraic numbers.

Intro 3: application of the binomial theorem 2

Moreover,
if $n = 0, 1, 2, \cdots$, then

$${}_1F_0(-n; -; -1) = \sum_{i=0}^{n} \frac{(-n, i)}{i!}(-1)^i = \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Remark 3:

- we can get combinatorial identities from the binomial theorem.

Intro 4: Aim(again)

Because ${}_1F_0$ can be expressed in terms of power functions,
we can

- do numerical calculation of algebraic numbers.
- get combinatorial identities.

Analogically,
if HGS can be expressed in terms of well-known functions,
then it is expected that those identities have many applications.

In this talk, we present a new systematic method for
finding HGS which can be expressed in terms of well-known functions.
Its method is called "the method of contiguity relations".

A proof of the binomial theorem 1

For finding our new method,
we continue to observe the binomial theorem.

Although general series can not be expressed in other functions,
${}_1F_0(a;-;x)$ was able to expressed using a power function.
From this, ${}_1F_0(a;-;x)$ must have "good property".

We shall make this "good property" clear
for finding our new method.

A proof of the binomial theorem 2

Remarking $x\frac{d}{dx}x^i = ix^i$, we get

$$\begin{aligned}
\left(x\frac{d}{dx}+a\right){}_1F_0(a;-;x) &= \sum_{i=0}^{\infty}\frac{(a,i)}{i!}\left(x\frac{d}{dx}+a\right)x^i \\
&= \sum_{i=0}^{\infty}\frac{(a,i)}{i!}(a+i)x^i = a\sum_{i=0}^{\infty}\frac{(a+1,i)}{i!}x^i \\
&= a{}_1F_0(a+1;-;x).
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}{}_1F_0(a;-;x) &= \sum_{i=1}^{\infty}\frac{(a,i)}{i!}ix^{i-1} \\
&= \sum_{i=0}^{\infty}\frac{(a,i+1)}{i!}x^i = a\sum_{i=0}^{\infty}\frac{(a+1,i)}{i!}x^i \\
&= a{}_1F_0(a+1;-;x).
\end{aligned}$$

A proof of the binomial theorem 3

The above two formulas lead to

$$ {}_1F_0(a;-;x) = (1-x)\,{}_1F_0(a+1;-;x). $$

Namely,

$$ \frac{{}_1F_0(a+(n-1);-;x)}{{}_1F_0(a+n;-;x)} = (1-x) \quad (n \in \mathbb{N}). \tag{1} $$

From this, we find the following "good property":

- two ${}_1F_0$ with the same parameter a up to additive integers are linearly related.
  (This linear relation can be regarded as the 1st order difference equation with respect to n).

The 1st order difference equation is expected to solve. →Let us try!

We substitute $n = 1, 2, \cdots, n$ into (1), multiply all of these:

$$ \frac{{}_1F_0(a;-;x)}{{}_1F_0(a+n;-;x)} = (1-x)^n. $$

A proof of the binomial theorem 4

Substituting $a = -n$ into above, we get

$$ {}_1F_0(-n; -; x) = (1-x)^n. \tag{2} $$

Although (2) holds for $n = 0, 1, 2, \cdots$,
we find (2) holds for $n \in \mathbb{C}$ thanks to Carlson's theorem.

Here, Carlson's theorem is follows:

$$ \begin{cases} f(n), g(n) \text{ are holomorphic for } \Re(n) \geq 0, \\ f(n), g(n) \text{ are } O(e^{k|n|}) \text{ for } \Re(n) \geq 0, \text{ where, } k < \pi, \\ f(n) = g(n) \ (n = 0, 1, 2, \cdots) \end{cases} $$
$$ \Rightarrow f(n) = g(n) \ (\Re(n) \geq 0). $$

Thus, we were able to get the binomial theorem.

What was "good property"?

We summarize a proof of the binomial theorem.
${}_1F_0(a;-;x)$ had "good property":

- two ${}_1F_0$ with the same parameter a up to additive integers are linearly related:

$$\frac{{}_1F_0(a+(n-1);-;x)}{{}_1F_0(a+n;-;x)} = (1-x) \quad (n \in \mathbb{N}). \tag{1}$$

This can be regarded as the 1st order difference equation wrt n.

Thanks to this good property,
we were able to get the value of ${}_1F_0(a;-;x)$.

$\longrightarrow$ In much the same way as the above method,
we can find HGS whose values can be obtained.
$\longrightarrow$Let us try.

Preparation 1: the Gauss hypergeoemtric series

The Gauss hypergeometric series(GHGS) is defined by

$$ {}_2F_1(a,b;c;x) := \sum_{i=0}^{\infty} \frac{(a)_i(b)_i}{(c)_i i!} x^i \quad (|x|<1). $$

We call

$$ \begin{aligned} (a,b,c) &: \text{ parameters}, \\ x &: \text{ independent variable}. \end{aligned} $$

Preparation 2: contiguity relations of GHGS

GHGS has the following property(contiguity relations):

For a given $(k, l, m) \in \mathbb{Z}^3$,
there exists a unique pair $(Q, R) \in (\mathbb{Q}(a, b, c, x))^2$ such that

$$
\begin{aligned}
&{}_2F_1(a+k, b+l; c+m; x) \\
&\qquad = Q\,{}_2F_1(a+1, b+1; c+1; x) + R\,{}_2F_1(a, b; c; x). \quad (3)
\end{aligned}
$$

Important thing :
for a given $(k, l, m) \in \mathbb{Z}^3$ coefficients Q, R can be exactly computed.

Now, we construct the method of contiguity relations.

The method of contiguity relations 1

(k, l, m):fix. We put
$(a, b, c) \to (a + (n-1)k, b + (n-1)l, c + (n-1)m)$ in (3):

$$\begin{aligned}
&{}_2F_1(a + nk, b + nl; c + nm; x) \\
&= Q^{(n)} {}_2F_1(a + (n-1)k + 1, b + (n-1)l + 1; c + (n-1)m + 1; x) \\
&+ R^{(n)} {}_2F_1(a + (n-1)k, b + (n-1)l, c + (n-1)m; x),
\end{aligned}$$

where

$$\begin{aligned}
Q^{(n)} &:= Q|_{(a,b,c)\to(a+(n-1)k, b+(n-1)l, c+(n-1)m)}, \\
R^{(n)} &:= R|_{(a,b,c)\to(a+(n-1)k, b+(n-1)l, c+(n-1)m)}.
\end{aligned}$$

Let (a, b, c, x) be a solution of the system

$$Q^{(n)} = 0 \quad \text{for } n = 1, 2, 3, \ldots. \tag{4}$$

The method of contiguity relations 2

Then, we get the 1st order difference equation wrt n:

$$\frac{{}_2F_1(a+(n-1)k, b+(n-1)l, c+(n-1)m; x)}{{}_2F_1(a+nk, b+nl; c+nm; x)} = \frac{1}{R^{(n)}}.$$

This is indeed "good property" stated before.
The above leads to

$$\frac{{}_2F_1(a, b; c; x)}{{}_2F_1(a+nk, b+nl; c+nm; x)} = \frac{1}{\prod_{i=1}^{n} R^{(i)}}.$$

From this, we can get values of ${}_2F_1(a, b; c; x)$.

These are the method of contiguity relations.

Summarization of the mehod of contiguity relations

Step 1: Give $(k, l, m) \in \mathbb{Z}^3$.

Step 2: Compute coefficients Q, R satisfying

$$
\begin{aligned}
&{}_2F_1(a+k, b+l; c+m; x) \\
&\qquad = Q\,{}_2F_1(a+1, b+1; c+1; x) + R\,{}_2F_1(a, b; c; x).
\end{aligned}
$$

Step 3: Find (a, b, c, x) satisfying

$$Q^{(n)} = 0 \quad \text{for } n = 1, 2, 3, \ldots \tag{4}$$

For this, since (the numerator of $Q^{(n)}$) $\in \mathbb{Z}[a, b, c, x][n]$, we only have to find (a, b, c, x) such that eliminate all of the coefficients of this polynomial wrt n.

Step 4: We get the 1st order difference equation wrt $n \in \mathbb{Z}$:

$$\frac{{}_2F_1(a+(n-1)k, b+(n-1)l, c+(n-1)m; x)}{{}_2F_1(a+nk, b+nl; c+nm; x)} = \frac{1}{R^{(n)}}.$$

By solving this, we can get values of ${}_2F_1(a, b; c; x)$.

Ex (1)： the case of $(k, l, m) = (0, 1, 1)$

$${}_2F_1\left(a, b+1; c+1; x\right) = Q\,{}_2F_1\left(a+1, b+1; c+1; x\right) + R\,{}_2F_1\left(a, b; c; x\right),$$

where

$$Q = \frac{a(1-x)}{a-c},\ R = \frac{c}{c-a}.$$

Therefore, for satisfying (4),

$$Q^{(n)} = \frac{a(1-x)}{a-c-n+1}$$

must be 0 regardless n. $\longrightarrow$ $(a, b, c, x) = (a, b, c, 1)$.
In this case, we have the 1st order difference equation

$$\frac{{}_2F_1(a, b+n-1; c+n-1; 1)}{{}_2F_1(a, b+n; c+n; 1)} = \frac{1}{R^{(n)}} = \frac{c-a+n-1}{c+n-1}.$$

We substitute $n = 1, 2, \cdots, n$ into this, multiply all of these:

Ex (2)： the case of $(k,l,m)=(0,1,1)$

$$\frac{{}_2F_1(a,b;c;1)}{{}_2F_1(a,b+n;c+n;1)}=\frac{(c-a)_n}{(c)_n}.$$

Substituting $b=-n(n\in\mathbb{Z}_{\geq 0})$ into the above, we obtain

(Chu-Vandermonde) $\quad {}_2F_1(a,-n;c;1)=\dfrac{(c-a)_n}{(c)_n}.$

The right hand side of the above is expressed as

$$\frac{(c-a)_n}{(c)_n}=\frac{\Gamma(c)\Gamma(c-a+n)}{\Gamma(c-a)\Gamma(c+n)}.$$

By using Carlson's theorem, and replacing n with $-b$, we yield

(Gauss) $\quad {}_2F_1(a,b;c;1)=\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$

$(\Re(c-a-b)>0)$.

Other examples

In the same manner, we can get many hypergeometric identities.

For exmaple...
In the case of $(k, l, m) = (-1, 0, 1)$, we obtain

(Kummer) $${}_2F_1(a, b; b+1-a; -1) = \frac{\Gamma(\frac{b}{2}+1)\Gamma(b+1-a)}{\Gamma(b+1)\Gamma(\frac{b}{2}+1-a)}.$$

In the cse of $(k, l, m) = (3, 2, 2)$, we get

$${}_2F_1\left(3a, 2a; 2a+\frac{1}{2}; \frac{1}{4}\right) = \frac{2^{6a+\frac{1}{2}}\Gamma(a+\frac{1}{4})\Gamma(a+\frac{3}{4})}{3^{3a+\frac{1}{2}}\Gamma(a+\frac{1}{3})\Gamma(a+\frac{2}{3})}.$$

By applying our method to ${}_2F_1$...

We found that
almost all known hypergeometric identities for ${}_2F_1$ can be obtained by applying the method of contiguity relations.
Of course, we were able to find many new identities.

Those identities has been listed in
Ebisu, A, *Special values of the hypergeometric series*, Mem. Amer. Math. Soc., (to appear).
also available arXiv:1308.5588.

Applying our method to other HGS

We can also make use of our method to have hypergeometric identities for

- the generalized HGS ${}_{p+1}F_p$
- Appell HGS F_1, F_2, F_3, F_4
- Lauricella HGS F_A, F_B, F_C, F_D

because these series have contiguity relations.

In particular,
identities for Appell-Lauricella HGS have not been studied.
Therefore, our method will become a potent tool
for investigating identities for many kinds of HGS,
especially Appell-Lauricella HGS.