On (q, h)–Weyl Algebras

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We introduce (q, h)–deformation of the Weyl algebra and show that the q-deformed universal enveloping algebra $U_q(sl(2, \mathbb{C}))$ is embedded into the tensor product of two $(q; h)$-Weyl algebras.

The talk is based on the following papers:

- S. Hilger, G. Filipuk, *Algebra embedding of U_q(sl(2)) into the tensor product of two (q, h)-Weyl algebras*, submitted.

- G. Filipuk, S. Hilger, *A remark on the tensor product of two (q, h)–Weyl algebras*, submitted.

The Classical Weyl Algebra

The classical Weyl algebra $\mathcal{W} = \mathcal{W}_{(1,0)}$ is the $\mathbb{C}$–algebra with a unit 1 generated by two elements D and X with a single relation

$$DX - XD \ = \ 1. \tag{1}$$

There is a standard representation on the space $\mathcal{C}^\infty(\mathbb{R},\mathbb{C})$ (or polynomials, analytic and other functions) by differentiation (operator D) and multiplication by the variable x (operator X). Indeed, (1) becomes

$$(xf(x))' - xf'(x) = f(x).$$

(q,h)–Deformed Weyl Algebra

Let $q > 0$ and $h \in \mathbb{R}$ be fixed real numbers. We do not consider the cases $q \in \mathbb{C}$, q is a unit root and so on.

The deformed (q,h)-Weyl algebra $\mathcal{W}_{(q,h)}$ is defined as the unital algebra over $\mathbb{C}$, generated by the operators

$$X,\ S,\ S^{-1},\ \mu^{-1}$$

subject to the relations

$$\begin{aligned} SX &= (qX+h)S, & (2)\\ SS^{-1} &= S^{-1}S = 1, & (3)\\ ((q-1)X+h)\mu^{-1} &= \mu^{-1}((q-1)X+h) = 1. & (4) \end{aligned}$$

The first relation (2) is the essential one.

In the limit $q \to 1$, $h \to 0$ we get $\mathcal{W}_{(1,0)}$.

Representations of the (q,h)–Deformed Weyl Algebra

I). Let $\sigma(x) = qx + h$, $x_0 \neq \frac{-h}{q-1}$.

The (q,h)–grid is

$$\mathbb{T}_{(q,h,x_0)} = \{\sigma^j(x_0) \,|\, j \in \mathbb{Z}\} \subseteq \mathbb{R}.$$

The representation on the space $\mathcal{F}(\mathbb{T}_{(q,h,x_0)}, \mathbb{C})$ of functions defined on the (q,h)–grid is as follows:

$$\begin{aligned} X f(x) &:= x\, f(x), \\ S f(x) &:= f(\sigma(x)) = f(qx+h), \\ S^{-1} f(x) &:= f(\sigma^{-1}(x)) = f(\rho(x)) = f(\tfrac{x-h}{q}), \\ \mu(X) f(x) &:= ((q-1)x + h) \cdot f(x), \\ \mu(X)^{-1} f(x) &:= ((q-1)x + h)^{-1} \cdot f(x). \end{aligned} \tag{5}$$

We have

$$SXf(x) = (qx+h)f(qx+h) = (qX+h)Sf(x).$$

II). The second representation of the (q,h)–Weyl algebra is on a suitable subspace of $\mathcal{C}^\infty(\mathbb{R},\mathbb{C})\cap\mathcal{L}^1(\mathbb{R},\mathbb{C})$, the space of infinitely differentiable and Lebesgue integrable functions. One can choose the Schwartz space of rapidly decreasing functions or $\mathcal{C}^\infty$ functions with compact support. Note that a function f of this type fulfills

$$\lim_{x\to-\infty} f(x) = 0, \qquad \lim_{x\to+\infty} f(x) = 0.$$

In this representation the operators act as

$$\begin{aligned} Xf(x) &:= i\cdot f'(x), \\ Sf(x) &:= e^{i\frac{h}{q}x} f(\tfrac{x}{q}), \\ S^{-1}f(x) &:= e^{-ihx} f(qx), \\ \mu(X)f(x) &:= (q-1)if'(x) + hf(x), \\ \mu(X)^{-1}f(x) &:= \tfrac{-i}{q-1}\int_{-\infty}^{x} e^{\frac{ih(x-t)}{q-1}} f(t)\,dt. \end{aligned} \tag{6}$$

One can easily check the fundamental relations of the (q,h)–Weyl algebra:

$$SX = (qX+h)S, \tag{7}$$

$$SS^{-1} = S^{-1}S = 1, \tag{8}$$

$$((q-1)X+h)\mu^{-1} = \mu^{-1}((q-1)X+h) = 1. \tag{9}$$

Various Relations in the (q,h)–Weyl Algebra

We can derive many additional relations for the various operators in the (q,h)–Weyl algebra.

For $j,k,\ell \in \mathbb{Z}$ the following general relations hold:

$$\begin{aligned} S^\ell \sigma^{j-\ell}(X) - \sigma^k(X) S^\ell &= ([j]_q - [k]_q)\mu(X) S^\ell, && (10)\\ S^\ell \mu(X) &= q^\ell \mu(X) S^\ell, && (11)\\ S^\ell \mu(X)^{-1} &= q^{-\ell}\mu(X)^{-1} S^\ell, && (12)\end{aligned}$$

$$\begin{aligned} DS^\ell &= q^\ell S^\ell D, && (13)\\ D\sigma^{j-1}(X) - \sigma^k(X) D &= ([j]_q - [k]_q)\mu(X) D + q^{j-1}, && (14)\\ D\mu(X) &= q\mu(X) D + (q-1). && (15)\end{aligned}$$

Here

$$[j]_q \quad := \quad \frac{q^j - 1}{q-1}$$

and the operator D is the (q,h)-forward difference operator. In the first representation it is given by

$$Df(x) = \frac{f(qx+h) - f(x)}{(q-1)x + h}.$$

(q,h)-Weyl Algebras for $q \neq 1$

Let $q \neq 1$ be fixed. The (q,h)-Weyl algebras $\mathcal{W}_{(q,h)}$, $h \in \mathbb{R}$, are isomorphic and an isomorphism

$$\phi : \mathcal{W}_{(q,0)} \to \mathcal{W}_{(q,h)}$$

is given by the transformation of the operator X as follows:

$$\phi : \mathcal{W}_{(q,0)} \to \mathcal{W}_{(q,h)}, \quad \begin{cases} X & \mapsto \quad X' = X - \frac{h}{q-1}, \\ S & \mapsto \quad S' = S, \\ S^{-1} & \mapsto \quad (S')^{-1} = S^{-1}, \\ \mu^{-1} & \mapsto \quad (\mu^{-1})' = \mu^{-1}. \end{cases}$$

The (q,h)-Weyl Algebras for $q=1$

For $h \neq 0$ the $(1,h)$-Weyl algebras are isomorphic and an isomorphism

$$\varphi : \mathcal{W}_{(1,h)} \to \mathcal{W}_{(1,h')}$$

is given on the set of generators as follows:

$$\varphi : \mathcal{W}_{(1,h)} \to \mathcal{W}_{(1,h')}, \quad \begin{cases} X & \mapsto & X' & = & \frac{h'}{h}X, \\ S & \mapsto & S' & = & S, \\ S^{-1} & \mapsto & (S')^{-1} & = & S^{-1}, \\ \mu^{-1} & \mapsto & (\mu^{-1})' & = & \frac{h}{h'}\mu^{-1}. \end{cases}$$

The Tensor Product of two (q,h)-Weyl Algebras $\mathcal{W}_{(q,h)}^{\otimes 2}$

The tensor product $\mathcal{W}_{(q,h)}^{\otimes 2}$ is generated by eight operators

$$\begin{array}{cccc} S_x, & X, & S_x^{-1}, & \mu_x^{-1}, \\ S_y, & Y, & S_y^{-1}, & \mu_y^{-1}. \end{array}$$

The four operators in the first (second) line fulfill the relations

$$S_x X = (qX + h) S_x, \qquad S_y Y = (qY + h) S_y, \tag{16}$$

$$S_x S_x^{-1} = S_x^{-1} S_x \ = \ 1, \qquad S_x S_x^{-1} = S_x^{-1} S_x \ = \ 1, \tag{17}$$

$$((q-1)X + h)\mu_x^{-1} = \mu_x^{-1}((q-1)X + h) \ = \ 1, \tag{18}$$

$$((q-1)Y + h)\mu_y^{-1} = \mu_y^{-1}((q-1)Y + h) \ = \ 1. \tag{19}$$

and any x-operator commutes with any y-operator.

Any representation of the (q,h)-Weyl algebra $W_{(q,h)}$ on a space of complex-valued functions of one variable x gives rise to a representation of the tensor product $W_{(q,h)}^{\otimes 2}$ on the "corresponding" space of functions of two variables x, y. The operators act as "partial" operators.

The q-Deformed Universal Enveloping Algebra $U_q = U_q(\mathrm{sl}(2,\mathbb{C}))$

The q-deformed universal enveloping algebra $U_q = U_q(\mathrm{sl}(2,\mathbb{C}))$ of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is generated as a unital algebra over $\mathbb{C}$ by the elements

$$E, \qquad F, \qquad H, \qquad K, \qquad K^{-1}$$

with the relations

$$\begin{aligned}
KK^{-1} &= K^{-1}K = 1, && (20)\\
(q-q^{-1})H &= K - K^{-1}, && (21)\\
[K,E] &= (q^2-1)EK, && (22)\\
[K,F] &= (q^{-2}-1)FK, && (23)\\
[E,F] &= H, && (24)\\
[H,E] &= E(qK+q^{-1}K^{-1}), && (25)\\
[H,F] &= -F(q^{-1}K+qK^{-1}). && (26)
\end{aligned}$$

[Ch. Kassel, Quantum groups]. The algebra U_q has many applications.

Algebra Embedding of $U_q(\mathrm{sl}(2))$ into $\mathcal{W}_{(q,h)}^{\otimes 2}$

Let

$$D_x = \frac{q}{q+1}\mu_x^{-1}(S_x - S_x^{-1}), \qquad D_y = \frac{q}{q+1}\mu_y^{-1}(S_y - S_y^{-1}).$$

The embedding of the algebra U_q into the tensor product of two (q,h)-Weyl algebras,

$$\psi:\ U_q \hookrightarrow \mathcal{W}_{(q,h)}^{\otimes 2}$$

for $q \neq 1$, is given by the following formulas:

$$E \mapsto \tfrac{\mu_x}{q-1}D_y, \qquad F \mapsto \tfrac{\mu_y}{q-1}D_x, \qquad H \mapsto \tfrac{\mu_x}{q-1}D_xS_y^{-1} - \tfrac{\mu_y}{q-1}D_yS_x^{-1},$$
$$K \mapsto S_xS_y^{-1}, \qquad K^{-1} \mapsto S_yS_x^{-1}.$$

For $h = 0$ this embedding $U_q \hookrightarrow \mathcal{W}_{(q,0)}^{\otimes 2}$ has the following form:

$$\begin{aligned} E &\mapsto XD_y = \tfrac{XY^{-1}(S_y - S_y^{-1})}{q-q^{-1}}, \qquad F \mapsto YD_x = \tfrac{YX^{-1}(S_x - S_x^{-1})}{q-q^{-1}}, \\ H &\mapsto XD_xS_y^{-1} - YD_yS_x^{-1} = \tfrac{S_xS_y^{-1} - S_yS_x^{-1}}{q-q^{-1}}. \end{aligned} \tag{27}$$

Computational Aspects

To find the embedding computationally (which can easily be implemented in any computer algebra system, for instance, Mathematica (www.wolfram.com)) we can use the first representation of the (q, h)–Weyl algebra.

The algebra U_q is isomorphic to the algebra U'_q generated by E, F, K, K^{-1}, H with relations

$$KK^{-1} = K^{-1}K = 1,$$

$$KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2} F,$$

$$[E, F] = H,$$

$$(q - q^{-1})H = K - K^{-1},$$

$$[H, E] = q(EK + K^{-1}E),$$

$$[H, F] = -q^{-1}(FK + K^{-1}F).$$

We assume that

$$K = S_x S_y^{-1}, \quad K^{-1} = S_y S_x^{-1}$$

and

$$E = a(x)a_1(y)D_y, \quad F = b_1(x)b(y)D_x,$$

where the coefficients are to be determined. We want to show that for $h \neq 0$ we can take

$$a(x) = x + \frac{h}{q-1},\ b(y) = y + \frac{h}{q-1},\ a_1(y) = b_1(x) = 1. \tag{28}$$

We have

$$KE = g_1(x,y)EK, \ \ KF = g_2(x,y)FK,$$

where

$$g_1(x,y) = \frac{qa(\sigma(x))a_1(\rho(y))}{a(x)a_1(y)}, \ \ g_2(x,y) = \frac{b(\rho(y))b_1(\sigma(x))}{qb(y)b_1(x)}.$$

Next, $H = EF - g_3(x,y)FE$ where

$$H = \frac{q^2 a(x)(a(\rho(x)) - a(\sigma(x)))a_1(y)b(\rho(y))b_1(x)}{(1+q)^2\mu(x)\mu(y)a(\rho(x))}(S_y S_x^{-1} - S_x S_y^{-1})$$

and

$$g_3(x,y) = \frac{a(x)b(\rho(y))}{a(\rho(x))b(y)}.$$

Next,

$$g_4(x,y)H = K - K^{-1},$$

where

$$g_4(x,y) = -\frac{(q+1)^2\mu(x)\mu(y)a(\rho(x)))}{q^2a(x)(a(\rho(x))) - a(\sigma(x))a_1(y)b(\rho(y))b_1(x)}.$$

However, an additional condition appears:

$$\frac{b(\sigma(y))}{b(\rho(y))} = \frac{a(\sigma(x))}{a(\rho(x))}. \quad (29)$$

Further,

$$HE - g_5(x,y)EH = g_6(x,y)EK + f_7(x,y)K^{-1}E,$$

where the functions $g_6(x,y)$ and $g_7(x,y)$ can be expressed in terms of $g_5(x,y)$ (the expressions are cumbersome and we omit them). The second condition on the unknown coefficients is

$$\frac{a_1(\sigma(y))}{a_1(\rho(y))} = \frac{q^2 a(\rho(x))}{a(\sigma(x))}. \tag{30}$$

Similarly,

$$HF - g_8(x,y)FH = g_9(x,y)FK + g_{10}(x,y)K^{-1}F,$$

where the functions $g_9(x,y)$ and $g_{10}(x,y)$ can be expressed in terms of $g_8(x,y)$ (the expressions are cumbersome and we omit them). The third condition on the unknown coefficients is

$$q^2 a(x)a(\rho(x))(a(x) - a(\rho^2(x)))b_1(\rho(x)) \tag{31}$$

$$+a(\rho^2(x))a(\sigma(x))(a(x) - a(\sigma^2(x)))b_1(\sigma(x)) = 0.$$

If $a(\sigma(x)) = \text{const}\, a(\rho(x)) = 0$ and taking the Ansatz $a(x) = Ax + B$ we find that const $= q^2$ and $B = A\,h/(q-1)$. Finally, we have

$$E = \frac{\mu(x)}{q-1}D_y, \quad F = \frac{\mu(y)}{q-1}D_x, \quad H = \frac{q}{q^2-1}(S_x S_y^{-1} - S_y S_x^{-1})$$

and the embedding follows.

Thank you very much for your attention!