A Maillet type theorem for generalized power series

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A classical Maillet theorem

We consider an m-th order ODE

$$F(z, u, \delta u, \ldots, \delta^m u) = 0, \tag{1}$$

where F is a polynomial of $m+2$ variables, $\delta = z(d/dz)$.

Theorem (Maillet, 1903). *Any power series solution $\varphi = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}[[z]]$ of (1) is of some Gevrey order $1/k \geqslant 0$, that is,*

$$|c_n| \leqslant AB^n (n!)^{1/k}$$

for some $A, B > 0$.

In other words, the series

$$\sum_{n=0}^{\infty} \frac{c_n}{(n!)^{1/k}} z^n \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(1+\frac{n}{k})} z^n$$

converges near the origin.

Exact estimates for the Gevrey order $1/k$ of the formal power series solution φ were obtained by J.-P. Ramis, B. Malgrange, Y. Sibuya in terms of the Newton polygon $\mathcal{N} = \mathcal{N}(\varphi)$.

Newton polygon

Attach to the formal power series solution φ of the initial equation $F=0$ a linear diff. operator

$$L_\varphi = \sum_{i=0}^{m} a_i(z)\delta^i$$

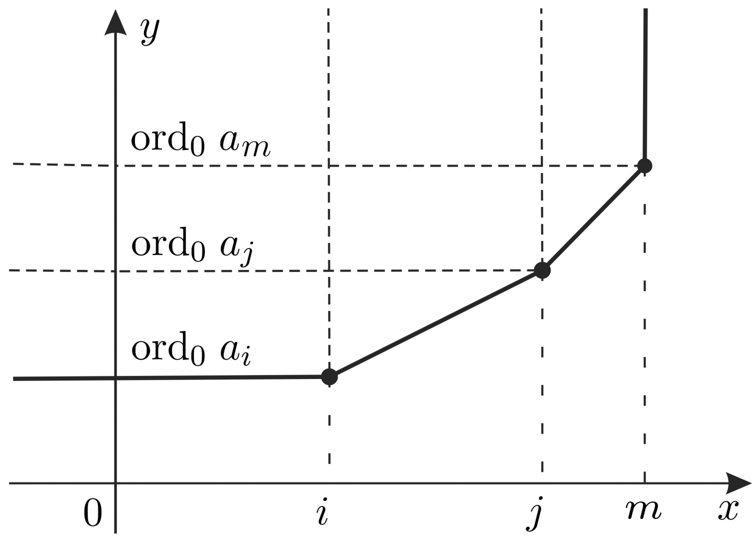
with the formal power series coefficients

$$a_i(z) = \frac{\partial F}{\partial u_i}(z, \varphi, \delta\varphi, \ldots, \delta^m\varphi) \in \mathbb{C}[[z]],$$

and draw on the (x,y)-plane all the sets

$$X_i = \{(x,y) \in \mathbb{R}^2 \mid x \leqslant i, y \geqslant \mathrm{ord}_0 a_i\}, \quad i = 0, 1, \ldots, m.$$

$$\mathcal{N} = \mathsf{hull}\Big(\bigcup_{i=0}^{m} X_i\Big)$$



The Newton polygon $\mathcal{N}$ with two positive slopes

$$k_1 = \frac{\mathrm{ord}_0\, a_j - \mathrm{ord}_0\, a_i}{j - i}, \quad k_2 = \frac{\mathrm{ord}_0\, a_m - \mathrm{ord}_0\, a_j}{m - j}.$$

Malgrange's theorem: the formal power series solution φ is of Gevrey order $1/k$, where k is the least of all the positive slopes of the Newton polygon $\mathcal{N}(\varphi)$ (or $k = +\infty$, if $\mathcal{N}(\varphi)$ has no positive slopes).

The refinement by Sibuya: φ is of the **exact** Gevrey order $1/k \in \{0, 1/k_1, \ldots, 1/k_s\}$, where $k_1 < \ldots < k_s$ are all of the positive slopes of $\mathcal{N}(\varphi)$.

Further development: Ramis–Sibuya asymptotic theorem, k-summability (multisummability) of formal power series solutions of ODEs (1990's).

Generalizations: problems of $\mathbb{M}$-summability for power series corresponding to strongly regular sequences $\mathbb{M}$ of non-Gevrey type (thus satisfying other types equations rather than ODEs) – lectures by Javier Sanz.

Generalized power series

We consider another generalization remaining within the framework of ODEs.

Let φ be a *generalized* power series of the form

$$\varphi = \sum_{n=0}^{\infty} c_n z^{s_n}, \qquad s_n \in \mathbb{C},$$

with the power exponents satisfying conditions

$$0 \leqslant \operatorname{Re} s_0 < \operatorname{Re} s_1 < \ldots, \qquad \lim_{n\to\infty} \operatorname{Re} s_n = +\infty.$$

We call $\operatorname{val} \varphi = s_0$ the *valuation* of φ.

For the generalized power series solution φ of the (polynomial) ODE

$$F(z, u, \delta u, \dots, \delta^m u) = 0$$

one may define the Newton polygon $\mathcal{N}(\varphi)$ in a similiar way as in the classical case. The only difference is that now

$$a_i(z) = \frac{\partial F}{\partial u_i}(z, \varphi, \delta\varphi, \dots, \delta^m \varphi)$$

are generalized power series, thus we draw on the (x, y)-plane the sets

$$X_i = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant i, y \geqslant \mathrm{Re}\,(\mathrm{val}\, a_i)\}.$$

Theorem 1. *Let* $\varphi = \sum_{n=0}^{\infty} c_n z^{s_n}$ *be a generalized power series solution of* $F = 0$ *and* k *the least of all the positive slopes of the Newton polygon* $\mathcal{N}(\varphi)$ (*or* $k = +\infty$, *if* $\mathcal{N}(\varphi)$ *has no positive slopes*). *Then*

$$|c_n| \leqslant AB^n \left|\Gamma(1 + \frac{s_n}{k})\right|$$

for some $A, B > 0$.

Steps of the proof

1) The existence of $\mu \in \mathbb{Z}_+$ such that all the exponents $s_n - s_\mu$, $n \geqslant \mu + 1$, belong to a finitely generated additive semi-group whose generators are linear independent over $\mathbb{Z}$. (Thus we are within the framework of so-called *grid-based* power series.)

2) The representation of generalized power series solutions by multivariate (Taylor) series.

3) The implicit mapping theorem for Banach spaces (of multivariate Taylor series).

An example

Consider an ODE

$$F = z^2 u\, \delta^3 u + (1 - 3z^2) u\, \delta^2 u - (\delta u)^2 + 2z^2 u\, \delta u + z^2 = 0.$$

It has a formal solution of the form

$$\varphi = c z^r + \sum_{n=0}^{\infty} c_n\, z^{s_n},$$

where $c \in \mathbb{C}^*$ is arbitrary, $\operatorname{Re} r < 1$ and

$$s_n \in r + \{l_1 + l_2(1 - r) \mid l_1, l_2 \in \mathbb{Z}_+, l_1 + l_2 > 0\}.$$

One has $\frac{\partial F}{\partial u_3}(z, \varphi, \ldots) = z^2 \varphi = c z^{r+2} + \ldots,$

hence $\mathsf{val}_3 = r+2$. The valuations of the other $\partial F/\partial u_i$ along φ are

$$\mathsf{val}_2 = \mathsf{val}_1 = \mathsf{val}_0 = r,$$

thus, the positive slope of $\mathcal{N}(\varphi)$ is

$$k = \frac{\mathrm{Re}(r+2) - \mathrm{Re}\, r}{3-2} = 2,$$

and one has $|c_n| \leqslant AB^n \left|\Gamma(1+s_n/2)\right|$.

Further possible questions

It seems to be natural to investigate some further questions concerning generalized power series solutions of ODEs:

- to refine Theorem 1 obtaining the **exact** Gevrey order;

- to think on the correctness of asymptotic Gevrey expansions and k-summability (Watson's lemma, Borel–Laplace transform, etc.);

- in the case of a positive answer to the previous item, to obtain some sufficient conditions for k-summability of solutions.