(q, h)-deformation of $U(sl(2))$

Stefan Hilger

Katholische Universität Eichstätt–Ingolstadt

Joint work with **Galina Filipuk**, University of Warsaw

Bedlewo — 15AAGA — September 17, 2015

Outline of Talk

A The (q, h)-deformed Weyl algebra — Definition

Noncommutative, associative and unital $\mathbb{C}$–algebra $\mathcal{W}_{(q,h)}$ with

Generators X, S $\qquad\qquad$ $\mu := (q-1)X + h$ (Graininess)

Relations

$$SX = (qX + h)S$$

S^{-1} exists

μ^{-1} exists

A The (q,h)-deformed Weyl algebra — Representation I

f defined on "Linear Graininess Time Scale"

$$
\begin{aligned}
Xf(x) &= x \cdot f(x) \\
Sf(x) &= f(qx+h) \\
S^{-1}f(x) &= f(\frac{x-h}{q}) \\
\mu f(x) &= [(q-1)x+h]\, f(x)
\end{aligned}
$$

A The (q, h)-deformed Weyl algebra — Representation II

On functions with compact support: $\mathcal{C}_c^\infty(\mathbb{R})$

$$\begin{aligned} Xf(x) &:= i \cdot f'(x) \\ Sf(x) &:= e^{i\frac{h}{q}x} f(\tfrac{x}{q}) \\ S^{-1}f(x) &:= e^{-ihx} f(qx) \\ \mu f(x) &:= (q-1)if'(x) + hf(x) \\ \mu^{-1}f(x) &:= \tfrac{-i}{q-1} \int_{-\infty}^{x} e^{\frac{ih(x-t)}{q-1}} f(t)\, dt \end{aligned}$$

A The (q,h)-deformed Weyl algebra — Representation III

On complex sequences: $\mathcal{F}(\mathbb{Z},\mathbb{C})$

$$\begin{aligned} Xf(x) &:= (q^x \cdot x_0 + h[x]_q)\, f(x) \\ Sf(x) &:= f(x+1) \\ S^{-1}f(x) &:= f(x-1) \\ \mu(X)f(x) &:= q^x \mu(x_0)\, f(x) \end{aligned}$$

A The (q,h)-deformed Weyl algebra — Diff Operators

Inside $\mathcal{W}_{(q,h)}$ define diff operators

$$D^+ := \mu^{-1}(S-1) \qquad \text{(Forward Diff)}$$

$$D^- := -\mu^{-1}(S^{-1}-1) \qquad \text{(Negative Backward Diff)}$$

Then

$$D^+X - XD^+ = 1 + \mu D^+$$

$$D^-X - XD^- = \frac{1}{q}(1-\mu D^-)$$

Other Presentation: Replace $\{X, S, S^{-1}, \mu^{-1}\} \to \{X, D^+, D^-, \mu^{-1}\}$

A The (q,h)-deformed Weyl algebra — Diff/Mix Operators

Inside $\mathcal{W}_{(q,h)}$ define

$$\begin{array}{lll} D & := & \frac{q}{q+1}\mu^{-1}(S - S^{-1}) \;=\; \frac{q}{q+1}(D^+ + D^-) \qquad \text{(Central Diff)} \\ M & := & \frac{q}{q+1}S + \frac{1}{q+1}S^{-1} \qquad \text{(Mix)} \end{array}$$

Then

$$DX - XD \;=\; M$$

$$MX - XM \;=\; \frac{q-1}{q}\mu M + \frac{1}{q}\mu^2 D$$

Third Presentation: Replace $\{X, S, S^{-1}, \mu^{-1}\} \to \{X, D, M, \mu^{-1}\}$

A The (q, h)-deformed Weyl algebra — Remark

Recall

$$SX = (qX + h)S$$

$$D^+ := \mu^{-1}(S - 1) = [(qX + h) - X]^{-1}(S - 1)$$

Then

$$qX + h \iff [D^+, \mathcal{P}_n(X)] = n \cdot \mathcal{Q}_{n-1}(X)$$

Applying the theory only works with the expression $qX + h$.

Mainly three cases: continuous — h-discrete — q-discrete

B The (q,h)-deformed Weyl algebra — Isomorphism I

For $q \neq 1$:

Weyl algebras $\mathcal{W}_{(q,0)}$ and $\mathcal{W}_{(q,h)}$ are isomorphic as rings.

$$\varphi : \mathcal{W}_{(q,0)} \to \mathcal{W}_{(q,h)} \qquad \begin{cases} X & \mapsto & X' & = & X - \frac{h}{q-1}, \\ S & \mapsto & S' & = & S, \\ S^{-1} & \mapsto & (S')^{-1} & = & S^{-1}, \\ \mu^{-1} & \mapsto & (\mu^{-1})' & = & \mu^{-1}. \end{cases}$$

B The (q,h)-deformed Weyl algebra — Isomorphism II

For $q = 1 \quad h \neq 0 \quad h' \neq 0$

Weyl algebras $\mathcal{W}_{(1,h)}$ and $\mathcal{W}_{(1,h')}$ are isomorphic as rings.

$$\varphi : \mathcal{W}_{(1,h)} \to \mathcal{W}_{(1,h')} \quad \begin{cases} X & \mapsto & X' & = & \frac{h'}{h} X, \\ S & \mapsto & S' & = & S, \\ S^{-1} & \mapsto & (S')^{-1} & = & S^{-1}, \\ \mu^{-1} & \mapsto & (\mu^{-1})' & = & \frac{h}{h'} \mu^{-1}. \end{cases}$$

B The (q, h)-deformed Weyl algebra — Embedding

For $h = 0 \quad q \neq 1 \quad q^j \neq 1$

Embedding (as rings) of $\mathcal{W}_{(q,0)}$ into $\mathcal{W}_{(q^j,0)}$.

$$\varphi : \mathcal{W}_{(q,0)} \to \mathcal{W}_{(q^j,0)} \qquad \begin{cases} X & \mapsto & X' & = & X^j, \\ S & \mapsto & S' & = & S, \\ S^{-1} & \mapsto & (S')^{-1} & = & S^{-1}, \\ \mu^{-1} & \mapsto & (\mu^{-1})' & = & \frac{(q-1)^j}{q^j-1}(\mu^{-1})^j. \end{cases}$$

C The Algebra U_q

(= basic nontrivial quantum group)
Unital $\mathbb{C}$–algebra U_q with

Generators E, F, K $\quad$ K^{-1} exists, $\quad$ $H := \frac{K-K^{-1}}{q-q^{-1}}$

Relations

$$\begin{aligned}
[K,E] &= (q^2-1)EK \\
[K,F] &= (q^{-2}-1)FK \\
[E,F] &= H \\
[H,E] &= E(qK+q^{-1}K^{-1}) \\
[H,F] &= -F(q^{-1}K+qK^{-1})
\end{aligned}$$

$\boxed{\text{C}}$ The Algebra U_q — Limit $q \to 1$

Degeneration to the universal embedding algebra $U(sl(2,\mathbb{C}))$ with

Generators E, F, H

Relations

$$[E,F] = H \qquad [H,E] = 2E \qquad [H,F] = -2F$$

Other generators

$$L_x = \tfrac{E-F}{2} \qquad L_y = i\tfrac{E+F}{2} \qquad L_z = \tfrac{i}{2}H$$

with relations

$$[L_x, L_y] = L_z \qquad [L_y, L_z] = L_x \qquad [L_z, L_x] = L_y$$

$\to$ Angular momentum of the Hydrogen atom

D Ladder

$$\cdots \qquad V_{n-1} \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} V_n \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} V_{n+1} \qquad \cdots$$

$$\alpha_n \qquad\qquad \alpha_{n+1}$$

A_n^+ Creation (Ascending / Raising) Linear Operators

A_n^- Annihilation (Descending / Lowering) Linear Operators

Given the above ladder and a number sequence α_n, define

$A_n^{\sqsupset}$	$:=$	$A_{n+1}^- A_{n+1}^+$	Right loop			
$A_n^{\sqsubset}$	$:=$	$A_n^+ A_n^-$	Left loop			
A_n^{Δ}	$:=$	$A_n^{\sqsupset} - A_n^{\sqsubset}$	Commutator	α_n^{Δ}	$:=$	$\alpha_{n+1} - \alpha_n$
$A_n^{\Diamond}$	$:=$	$\frac{A_n^{\sqsupset} + A_n^{\sqsubset}}{2}$	Anticommutator	$\alpha_n^{\Diamond}$	$:=$	$\frac{\alpha_{n+1} + \alpha_n}{2}$

D Ladder Homomorphism

$$
\begin{array}{ccccccc}
\cdots & V_{n-1} & \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} & V_n & \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} & V_{n+1} & \cdots \\
& \downarrow \Phi_{n-1} & & \downarrow \Phi_n & & \downarrow \Phi_{n+1} & \\
\cdots & W_{n-1} & \underset{B_n^-}{\overset{B_n^+}{\rightleftarrows}} & W_n & \underset{B_{n+1}^-}{\overset{B_{n+1}^+}{\rightleftarrows}} & W_{n+1} & \cdots
\end{array}
$$

$(\Phi_n)_{n\in\mathbb{Z}}$ Ladder homomorphism

$\iff$ All Squares commute

$$
\iff \quad \begin{cases} \Phi_{n+1}A_{n+1}^+ = B_{n+1}^+\Phi_n \\ \Phi_{n-1}A_n^- = B_n^-\Phi_n \end{cases} \quad \text{for all } n.
$$

D Ladders — Subspaces

$$\cdots \qquad V_{n-1} \underset{\substack{A_n^- \\ \alpha_n}}{\overset{A_n^+}{\rightleftarrows}} V_n \underset{\substack{A_{n+1}^- \\ \alpha_{n+1}}}{\overset{A_{n+1}^+}{\rightleftarrows}} V_{n+1} \qquad \cdots$$

Now, given a ladder, define sequence of subspaces

$$\begin{aligned} V_n' &:= \operatorname{eig}(A_n^{\sqsupset}, \alpha_{n+1}) \cap \operatorname{eig}(A_n^{\sqsubset}, \alpha_n) \\ &= \operatorname{eig}(A_n^{\Delta}, \alpha_n^{\Delta}) \cap \operatorname{eig}(A_n^{\Diamond}, \alpha_n^{\Diamond}) \end{aligned}$$

Is this a subladder (with operators still well–defined)?

$$\cdots \qquad V'_{n-1} \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} V'_n \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} V'_{n+1} \qquad \cdots$$

D Ladders — Observation

IF

$(A_n^\Delta - \alpha_n^\Delta)$ is a ladder endomorphism

AND

$(A_n^\Diamond - \alpha_n^\Diamond)$ is a ladder endomorphism,

THEN

(V_n') is a subladder.

D Ladder Theorem

IF

$(A_n^\Delta - \alpha_n^\Delta)$ is a ladder endomorphism

OR

$(A_n^\Diamond - \alpha_n^\Diamond)$ is a ladder endomorphism,

THEN

(V_n') is a subladder.

X Symmetric q numbers

Let $q \in \mathbb{R}^+$. For $j \in \mathbb{R}$ we define

$$[\![j]\!] \;=\; [\![j]\!]_q \;:=\; \tfrac{q^j - q^{-j}}{q - q^{-1}}.$$

$$[\![j+k]\!] - [\![j-k]\!] \;=\; (q^j + q^{-j})[\![k]\!],$$

$$[\![j+k]\!][\![j-k]\!] \;=\; [\![j]\!]^2 - [\![k]\!]^2,$$

$$[\![j+k]\!]^2 + [\![j-k]\!]^2 \;=\; (q^{2k} + q^{-2k})[\![j]\!]^2 + 2[\![k]\!]^2.$$

E The structure ladder of U_q

$$\cdots V_{m-1} \underset{F}{\overset{-E}{\rightleftarrows}} V_m \underset{F}{\overset{-E}{\rightleftarrows}} V_{m+1} \cdots$$

$$A^{\Delta} = EF - FE = H = \frac{K - K^{-1}}{q - q^{-1}}$$

$$A^{\Diamond} = -\tfrac{1}{2}(EF + FE)$$

$$P = \tfrac{q+q^{-1}}{2} \, \tfrac{K-2+K^{-1}}{(q-q^{-1})^2} \qquad \alpha_m = [\![m - \tfrac{1}{2}]\!]^2 - [\![\ell + \tfrac{1}{2}]\!]^2$$

$$C = A^{\Diamond} - P$$

$$V_m' = \mathrm{Eig}(K, q^{2m}) \cap \mathrm{Eig}(C, -[\![\ell]\!][\![\ell+1]\!]) = \langle x^{\ell+m} \cdot y^{\ell-m} \rangle$$

E Embedding $U_q(sl(2,\mathbb{C})) \hookrightarrow \mathcal{W}_{(q,0)}^{\otimes 2}$

$$
\begin{aligned}
E &\mapsto XD_y \\
F &\mapsto YD_x \\
K &\mapsto S_x S_y^{-1} \\
K^{-1} &\mapsto S_y S_x^{-1}
\end{aligned}
$$

$$
\cdots V_{m-1} \underset{YD_x}{\overset{-XD_y}{\rightleftarrows}} V_m \underset{YD_x}{\overset{-XD_y}{\rightleftarrows}} V_{m+1} \cdots
$$

$\boxed{\textbf{F}}$ **Embedding** $U_q(sl(2,\mathbb{C})) \hookrightarrow \mathcal{W}_{(q,0)}^{\otimes 2}$ — **Structure Ladder** $q \to 1$

$$
\begin{aligned}
P &= \tfrac{q+q^{-1}}{2}\tfrac{q^2}{(q+1)^2}(XD_x^+ - YD_y^+)(XD_x^- - YD_y^-) \\
&\xrightarrow{q\to 1} \tfrac{1}{4}(XD_x - YD_y)^2 \;=\; \tfrac{1}{4}H^2
\end{aligned}
$$

$$
\begin{aligned}
C &= -\tfrac{q^2}{(q+1)^2}\,[q^{\frac{1}{2}}XD_x^+ + q^{-\frac{1}{2}}YD_y^- + q^{-\frac{1}{2}}] \\
&\qquad\qquad [q^{\frac{1}{2}}YD_y^+ + q^{-\frac{1}{2}}XD_x^- + q^{-\frac{1}{2}}] + [\![\tfrac{1}{2}]\!]^2 \\
&\xrightarrow{q\to 1} -\tfrac{1}{4}(XD_x + YD_y + 1)^2 + \tfrac{1}{4}.
\end{aligned}
$$

F Embedding $U_q(sl(2,\mathbb{C})) \hookrightarrow \mathcal{W}_{(q,0)}^{\otimes 2}$ — Structure Ladder $q \to 1$

$$V_m' \;=\; \mathrm{Eig}(H, 2m) \;\cap\; \mathrm{Eig}(C, -\ell(\ell+1)) \;=\; \langle x^{\ell+m} \cdot y^{\ell-m} \rangle$$

R

References

[1] S. Hilger, G. Filipuk. *Factorization of* (q, h)*-difference operators – an algebraic approach.* J. Difference Eq. Appl. **20**, pp. 1201–1221 (2014).

[2] S. Hilger, G. Filipuk. *Algebra embedding of* $U_q(sl(2))$ *into the tensor product of two* (q,h)*-Weyl algebras.* Journal of Pure and Applied Algebra, Accepted. (2015)

[3] G. Filipuk, S. Hilger. *Linear graininess times scales and ladder operators of orthogonal polynomials.* Results Math. **64**, pp. 13–35 (2013)

[4] Ch. Kassel. *Quantum Groups.* Springer-Verlag, Berlin – Heidelberg – New York (1995)

[5] P.P. Kulish and N. Yu. Reshetikhin, *Quantum linear problem for the sine-Gordon equation and higher representations*, Zap. Nauchn. Sem. Leningrad. Otdel Mat. Inst. Steklov. 101 (1981), 101 - 110. English translation: J. Soviet Math. 23 (1983), 2435-2441.

Thank You!