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Ramified irregular singularities of meromorphic connections and plane curve singularities

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1 Motivation

- Understand ramified irregular singularities of local linear ODE.

$$\exp(\alpha_p x^{-\frac{p}{q}} + \alpha_{p-1} x^{-\frac{p-1}{q}} + \cdots) x^{\lambda} \phi(x) \qquad \phi(x) \in \mathbb{C}[\![x^{\frac{1}{q}}]\!]$$

(Hukuhara-Turrittin theory. cf. Newton-Puiseux theory for algebraic equations)

- Understand Stokes structures.

Deformation parameters for isomonodromic deformation

Fuchsian case

Configuration spaces of ordered sets of points on $\mathbb{P}^1$

$$\mathcal{F}_n(\mathbb{P}^1) = \{(u_1, \ldots, u_n) \in \prod_{i=1}^{n} \mathbb{P}^1 \mid u_i \neq u_j \text{ for all } i \neq j\}$$

$$\mathcal{F}_{3,n}(\mathbb{P}^1) = \mathcal{F}_n(\mathbb{P}^1 \backslash \{0, 1, \infty\})$$

Understand deformation manifolds for irregular singular cases.

Today's tool

irregular singularity	**plane curve singularity**
Komatsu-Malgrange irregularity	Milnor number intersection number
local Fourier transform	blow up
deformation manifold Stokes structure	knot (link) structure

2 Milnor number & irregularity

$q \in \mathbb{Z}_{>0}$, $f \in \mathbb{C}[x^{-\frac{1}{q}}]$, $\deg_{x^{-\frac{1}{q}}} f = p$.
$E_{f,q}$: fin. dim. $\mathbb{C}(\!(x)\!)\langle\partial\rangle$-module
$E_{f,q} := \mathbb{C}(\!(x^{\frac{1}{q}})\!)$ as $\mathbb{C}(\!(x)\!)$-vec. sp.,
$\partial\, m := (\frac{d}{dx} + x^{-1} f)\, m$ for $m \in \mathbb{C}(\!(x^{\frac{1}{q}})\!)$

Remark 2.1. *$E_{f,q}$ represents local linear ODE with formal solutions*

$$\phi(x)e^{F(x)},$$

$\phi(x) \in \mathrm{GL}(q, \mathbb{C}(\!(x^{\frac{1}{q}})\!))$, $\overline{f} = \int f/x\, dx$,
$F(x) = \mathrm{diag}\{\overline{f}(x), \overline{f}(e^{2\pi\sqrt{-1}}x), \ldots, \overline{f}(e^{2\pi\sqrt{-1}(q-1)}x)\}$

Remark 2.2. *From the Hukuhara-Turrittin theory of formal differential equations, $E_{f,q}$ represent all irreducible fin. dim. $\mathbb{C}(\!(x)\!)\langle\partial\rangle$-modules.*

For irreducible $E_{f,q}$, we attach the following germ of plane algebraic curve.

$$C_{f,q} := \prod_{i=1}^{q} \left(y - \frac{1}{f(\zeta_q^i x^{\frac{1}{q}})} \right) \in \mathbb{C}[\![x, y]\!],$$

$\zeta_q := e^{2\pi\sqrt{-1}/q}$.

Komatsu-Malgrange irregularity

M: fin. dim. $\mathbb{C}((x))\langle\partial\rangle$-module.

$$M \cong \bigoplus_{i=1}^{n} (E_{f_i,q_i} \otimes J_{m_i}) \quad \text{(HTL indecomp. decomp.)}$$

$$\mathrm{Irr}(M) = \sum_{i=1}^{n} q_i \mathrm{deg}(f_i) \quad \text{(Komatsu-Malgrange irregularity)}$$

$\mathrm{Irr}(M)$ is important analytic invariant. For example,

$$\mathrm{Irr}(\mathrm{End}_{\mathbb{C}((x))} M) \sim$$
$$\mathrm{dim}\{\text{formal isom. class of } M\}/\{\text{merom. isom. class}\}$$

Intersection number & Milnor number

$f, g \in \mathbb{C}[\![x, y]\!]$: irreducible plane curve germs

$$I(f,g) := \dim_{\mathbb{C}} \mathbb{C}[\![x,y]\!]/\langle f, g\rangle \quad \text{(intersection number)},$$

$$\mu(f) := I\left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f\right) \quad \text{(Milnor number)}$$

Milnor number is a topological invariant. For example,

μ = 1st Betti number of Milnor fiber of the curve,

= degree of Alexander poly. of the knot of the curve,

$= \cdots$

Theorem 2.3. $E_{f,q}$, $E_{g,q'}$*: irred. non-isomorphic,* $\mathrm{ord}(f) = -p/q$, $\mathrm{ord}(g) = -p'/q'$.

$$I(C_{f,q}, C_{g,q'}) = pq' + p'q - \mathrm{Irr}(\mathrm{Hom}_{\mathbb{C}((x))}(E_{f,q}, E_{g,q'}))$$

Corollary 2.4. μ*: Milnor number of* $C_{f,q}$

$$\mu = (2p-1)(q-1) - \mathrm{Irr}(\mathrm{End}_{\mathbb{C}((x))}(E_{f,q})).$$

Analytic inv. of $E_{f,q}$ $\longleftrightarrow$ Topological inv. of $C_{f,q}$

3 Local Fourier transform & blowing up

Definition 3.1 (Local Fourier transform $\mathcal{F}^{(0,\infty)}$, (Laumon, Bloch-Esnault, Garcia-Lopez)). *Suppose* $\operatorname{ord}(f) < 0$.
$(\operatorname{ord}(\sum_{i=-\infty}^{\infty} a_i x^i) := \min\{i \mid a_i \neq 0\})$
$\widetilde{E}_{f,q}^{(0,\infty)} := \mathcal{F}^{(0,\infty)}(E_{f,q})$ *is defined as follows.*

$$\widetilde{E}_{f,q}^{(0,\infty)} := E_{f,q} \quad \text{as } \mathbb{C}\text{-vec. sp.}$$

$$\begin{cases} x_{\widetilde{E}}^{-1} m := -\partial_E m \\ (x^2\partial)_{\widetilde{E}} m := -x_E m \end{cases}, \qquad m \in \widetilde{E}_{f,q}^{(0,\infty)} = E_{f,q}$$

Remark 3.2.

$$\mathcal{F}^{(0,\infty)} : (x, \frac{d}{dx}) \xrightarrow{Fourier} (\frac{d}{dx}, -x) \xrightarrow{\zeta = \frac{1}{x}} (-\zeta^2 \frac{d}{d\zeta}, -\zeta^{-1})$$

Definition 3.3 ($\mathcal{F}^{(\infty,0)}$, $\mathcal{F}^{(\infty,\infty)}$). *(i) Suppose* $0 > \mathrm{ord}(f) > -1$. $\widetilde{E}_{f,q}^{(\infty,0)} = \mathcal{F}^{(\infty,0)}(E_{f,q})$ *is defined as follows.*

$$\widetilde{E}_{f,q}^{(\infty,0)} := E_{f,q} \qquad \text{as } \mathbb{C}\text{-vec. sp.}$$
$$\begin{cases} x_{\widetilde{E}} m := (x^2\partial)_E m \\ \partial_{\widetilde{E}} m := -x_E^{-1} m \end{cases}, \quad m \in \widetilde{E} = E.$$

(ii) Suppose $\mathrm{ord}(f) < -1$. $\widetilde{E}_{f,q}^{(\infty,\infty)} = \mathcal{F}^{(\infty,\infty)}(E_{f,q})$ *is defined as follows.*

$$\widetilde{E}_{f,q}^{(\infty,\infty)} := E_{f,q} \qquad \text{as } \mathbb{C}\text{-vec. sp.}$$
$$\begin{cases} x_{\widetilde{E}}^{-1} m := (x^2\partial)_E m \\ (x^2\partial)_{\widetilde{E}} m := -x_E^{-1} m \end{cases}, \quad m \in \widetilde{E} = E.$$

Theorem 3.4 (Garcia-Lopez). *M: holon. $\mathbb{C}[x]\langle \frac{d}{dx}\rangle$-mod.*
$\widehat{M}_c := \mathbb{C}((x_c))\langle \frac{d}{dx_c}\rangle \otimes M$, $x_c = x - c$,
$\widehat{M}_\infty := \mathbb{C}((x^{-1}))\langle \frac{d}{dx}\rangle \otimes_{\mathbb{C}[x^{-1}]\langle \frac{d}{dx}\rangle} M[x^{-1}]$.
M^F: Fourier transform of M.

$$\widehat{M^F}_0 \cong \mathcal{F}^{(\infty,0)}(\widehat{M}_\infty^{<1}),$$

$$\widehat{M^F}_\infty \cong \bigoplus_{c \in \mathrm{Sing} M \cap \mathbb{C}} \mathcal{F}^{(c,\infty)}(\widehat{M}_c) \oplus \mathcal{F}^{(\infty,\infty)}(\widehat{M}_\infty^{>1}).$$

Here $\widehat{M}_a^{\gtrless \alpha}$ stands for components with slope of Newton polygon $\gtrless \alpha$.

Blow up

Definition 3.5 (quadratic transform, blow up)**.**

$$\begin{array}{rccc} \sigma_1\,(\textit{resp. } \sigma_2)\colon & \mathbb{C}[\![x,y]\!] & \to & \mathbb{C}[\![x_1,y_1]\!] \\ & x & \mapsto & x_1\,(\textit{resp. } x_1y_1) \\ & y & \mapsto & x_1y_1\,(\textit{resp. } y_1) \end{array}$$

Remark 3.6. *Blow up of* $\mathbb{C}^2$ *at* $(0,0)$

$$\pi\colon T := \{(x,y,(\xi:\eta)) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid x\eta = y\xi\} \longrightarrow \mathbb{C}^2$$

$$\sigma_1^*\colon T \cap (\mathbb{C}^2 \times U_1) \longrightarrow \mathbb{C}^2, \quad U_1 := \{(\xi:\eta) \mid \xi \neq 0\} \cong \mathbb{C}$$

$$\sigma_2^*\colon T \cap (\mathbb{C}^2 \times U_2) \longrightarrow \mathbb{C}^2, \quad U_2 := \{(\xi:\eta) \mid \eta \neq 0\} \cong \mathbb{C}$$

Theorem 3.7. *Denote plane curve associated to $\mathcal{F}^{(*,*)}(E_{f,q})$ is written by $\widehat{C}^{(*,*)}(x,y)$.*

1. $$\widehat{C}^{(\infty,0)}(x_1,-y_1)=\sigma_2^*(C_{\tilde{f},q}(x,y)),$$
   $\tilde{f}=f+\frac{p}{2(q-p)}$.
2. $$C_{\tilde{f},q}(-x_1,y_2)=\sigma_2^*(\widehat{C}^{(0,\infty)}(x,y)),$$
   $\tilde{f}=f-\frac{q}{2(p+q)}$.
3. $$\widehat{C}^{(\infty,\infty)}(x_1,-y_1)=\sigma_3^*(C_{\tilde{f},q}(x,y)),$$
   $\tilde{f}=f+\frac{p}{2(p-q)}$.

$\underline{\sigma_3}$

$$\sigma_3 \colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x_1 = x^{-1}y \\ y_1 = y \end{pmatrix}$$

The above thm obtained by tracing the computation of local Fourier transforms by J. Fang and C. Sabbah.

As an application, we can consider "resolution of ramified irregular singularity" as an analogue of resolution of singularity (cf. M. Noether,..., H. Hironaka).

4 Deformation manifold, Stokes structure & knot structure

Knot & irred. plane curve singularity $f(x,y) \in \mathbb{C}[\![x,y]\!]$:
irreducible plane curve germ s.t.

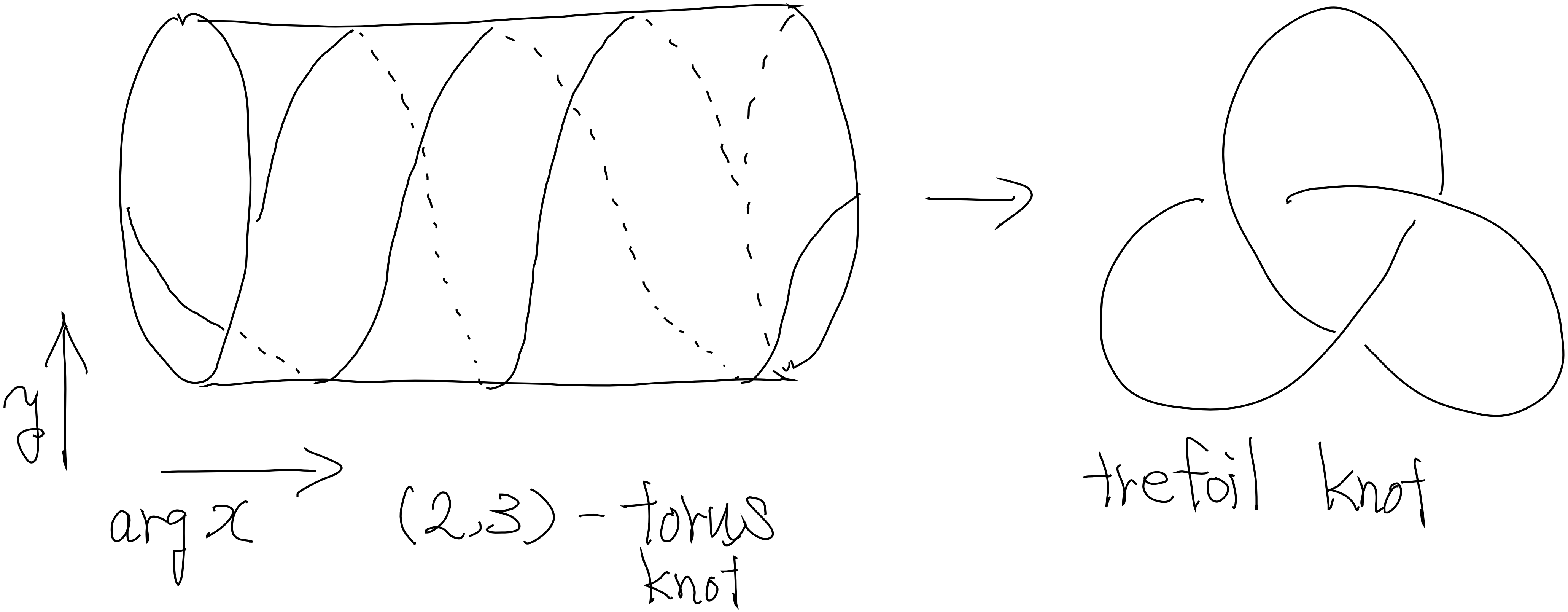
$$f(x,y) = \prod_{i=1}^{q} (y - \alpha(\zeta_q^i x^{\frac{1}{q}})), \quad \alpha(t) \in \mathbb{C}\{t\}.$$

$$K_f := \{(x(\tau), \alpha(x(\tau)^{\frac{1}{q}})) \mid x(\tau) = \eta e^{2\pi\sqrt{-1}\tau},\ \tau \in \mathbb{R}\},$$

$0 < \eta << 1$ (knot of plane curve f)

Example

$$f(x,y) = y^2 - x^3 = (y - x^{\frac{3}{2}})(y + x^{\frac{3}{2}})$$



<u>Deformation manifold</u>

Fuchsian case

$$\mathcal{F}_n(\mathbb{P}^1)$$

Non-Fuchsian case

Local formal solution

$$\phi(x^{\frac{1}{q}})x^L e^{F(x)}$$

$\phi(x^{\frac{1}{q}}) \in \mathrm{GL}_n(\mathbb{C}(\!(x^{\frac{1}{q}})\!))$, $L \in \mathrm{GL}_n(\mathbb{C})$,
$F(x) = \mathrm{diag}(f_1(x^{-\frac{1}{q}}), \ldots, f_n(x^{-\frac{1}{q}}))$, $f_i \in x^{-\frac{1}{q}}\mathbb{C}[x^{-\frac{1}{q}}]$
We move coefficients in $F(x)$ appropriately.

Precisely, set $n_{i,j} := \mathrm{ord}(f_i - f_j)$ and

$$\mathcal{M}_F := \{(g_i) \in \prod_{i=1}^{n} x^{-\frac{1}{q}}\mathbb{C}[x^{-\frac{1}{q}}] \mid \begin{array}{l} \mathrm{ord}(g_i - g_j) = n_{i,j}, \\ \mathrm{ord}(g_k) \geq \min(n_{i,j}) \end{array} \}$$

(local deformation manifold of $F(x)$)

Theorem 4.1. *$E_{f,q}$, $E_{f',q}$: irred., $\mathrm{ord}(f) = \mathrm{ord}(f')$ $E_{f,q}$ and $E_{f',q}$ define the same local deformation manifold if and only if the knots of $C_{f,q}$ and $C_{f',q}$ are equivalent.*

There will be a close connection between

- local deformation manifold,
- moduli space of plane curve germs with the fixed knot.

Stokes structure (Rough and Broken!!)

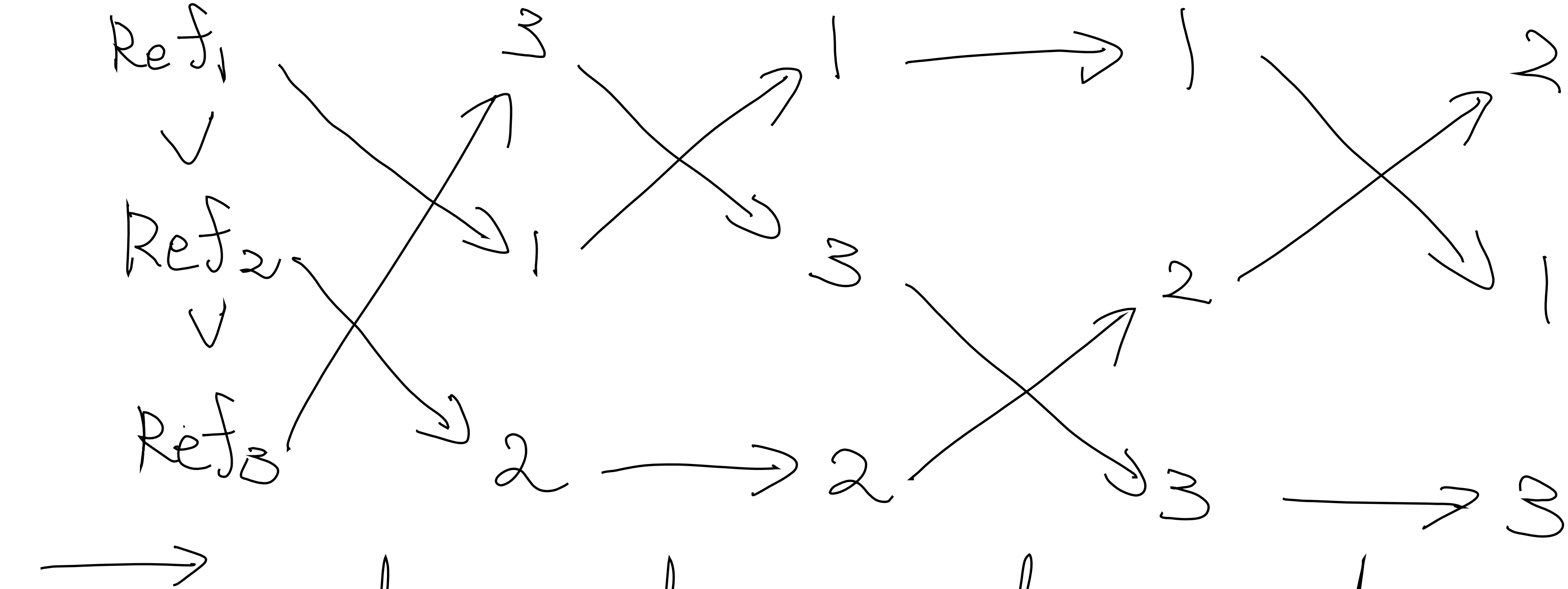
FIND invariants of Stokes structure of $E_{f,q}$
with the fixed deform. mfd. !

Roughly, Stokes phenomenon induced from the changes of orderings of

$$\mathrm{Re}(f_1(x^{-\frac{1}{q}})), \ldots, \mathrm{Re}(f_q(x^{-\frac{1}{q}})),$$

when x moves along a small circle around $x = 0$.

Seq. of permutations



$\arg x$ $\quad d_1 \quad d_2 \quad d_3 \quad d_4$: Stokes dir.

$s_1 \quad s_2 \quad s_3 \quad s_4 \;:\; \mathfrak{S}_3$

$\Longrightarrow$ (s_1, s_2, s_3, s_4) : seq. of elem. of $\mathfrak{S}_3$

As an analogy of curve singularities,

$$\tilde{K}_{f,q} = \{(x, y = f_i(x^{-\frac{1}{q}})) \mid x \in \text{small}\, S^1\}$$

draws a BIG knot.

$$\begin{array}{ccc} \tilde{K}_{f,q} & \xrightarrow{\mathrm{Re}(y)} & \{\mathrm{Re}(f_1), \ldots, \mathrm{Re}(f_q)\} \\ {\scriptstyle \text{normalise}}\downarrow & & \downarrow \\ \text{word of } B_q & \xrightarrow{B_q \to \mathfrak{S}_q} & \text{seq. of perm.} \end{array}$$

The detail can be found in "Local Fouirer transform and blowing up" as arXiv:1406.5788.