On the relationship between the BNR equation and the Pearcey system

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Outline of this talk

1. Motivation & Main results
2. Exact WKB analysis for the BNR equation
3. Exact WKB analysis for the Pearcey system
4. Relationship between the BNR equation & the Pearcey system

Motivation & Main Results (1/2)

The BNR equation (Berk-Nevins-Roberts, 1982):

$$\left(\eta^{-3}\frac{d^3}{dx_1^3}+\frac{1}{2}c\eta^{-1}\frac{d}{dx_1}+\frac{1}{4}x_1\right)\psi=0$$

Fact

The BNR equation has two types of Stokes curves:

- Ordinary Stokes curve (Stokes curve emanating from ordinary turning point)
- New Stokes curve (Stokes curve emanating from virtual turning point)

The BNR equation is given by the restriction of the Pearcey (皮爾西) system

$$\begin{cases}\left(\eta^{-3}\dfrac{\partial^3}{\partial x_1^3}+\dfrac{1}{2}x_2\eta^{-1}\dfrac{\partial}{\partial x_1}+\dfrac{1}{4}x_1\right)\psi=0,\\ \left(\eta^{-1}\dfrac{\partial}{\partial x_2}-\eta^{-2}\dfrac{\partial^2}{\partial x_1^2}\right)\psi=0\end{cases}$$

to $x_2=c$

- This system is a holonomic system (a completely integrable system) with rank 3
- The Pearcey integral $\int e^{\eta(t^4+x_2t^2+x_1t)}dt$ (Pearcey, 1946) satisfies the Pearcey system

Exact WKB analysis for holonomic system is initiated by Aoki (2005)
Shimomura consider a Pfaffian system containing parameters (1979)

Motivation & Main Results (2/2)

Main Results

- Ordinary Stokes curves and a new Stokes curve for the BNR equation are contained in the Stokes surface for the Pearcey system
  - Stokes surface: a counterpart of Stokes curve
- Every (singular perturbed) holonomic system with two independent variables can be transformed to the Pearcey system at a cuspidal singularity of the set of the turning points
- We give an algorithm for the construction of the Pearcey system from the BNR equation

Exact WKB analysis for the BNR equation (1/3)

The BNR equation:

$$\left(\eta^{-3}\frac{d^3}{dx_1^3}+\frac{1}{2}c\eta^{-1}\frac{d}{dx_1}+\frac{1}{4}x_1\right)\psi=0$$

Def: WKB solution, ordinary turning point, ordinary Stokes curve

- WKB solution is given by the following formal power series

$$\psi_j=\exp\left(\eta\int^{x_1}\omega_j\right)\sum_{n=0}^{\infty}\eta^{-(n+1/2)}\psi_{j,n}(x_1)$$

  where $\omega_j=\zeta_{1,j}(x_1)dx_1$ and $\zeta_{1,j}$ satisfies

$$p(x_1,\zeta_1)=\zeta_1^3+\frac{1}{2}c\zeta_1+\frac{1}{4}x_1=0$$

- A point $a\in\mathbb{C}$ is called a turning point if there exist two 1-forms ω_j and $\omega_{j'}$ such that $\omega_j(a)=\omega_{j'}(a)$, i.e., $\zeta_{1,j}(a)=\zeta_{1,j'}(a)$
- A Stokes curve emanating from $x=a$ is defined by

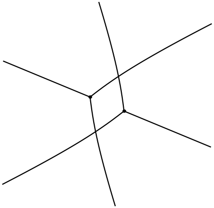
$$\Im\int_a^{x_1}(\omega_j-\omega_{j'})=0$$

Exact WKB analysis for the BNR equation (2/3)

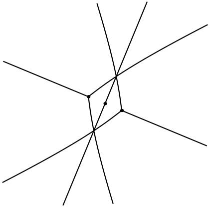
By the definition, the set of the turning points is given by the discriminant of $p(x, \zeta) = 0$, that is,

$$27x_1^2 + 8c^3 = 0$$

- The BNR equation has 6 ordinary Stokes curves
- Stokes phenomena occur on these ordinary Stokes curves



- Stokes phenomena occur also on the so-caled "new Stokes curve" (Berk-Nevins-Roberts, 1982)
- A new Stokes curve can be interplated as a Stokes curve emanating from the so-called "virtual turning point" (Aoki-Kawai-Takei, 1994)



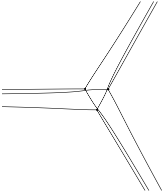
Exact WKB analysis for the BNR equation (3/3)

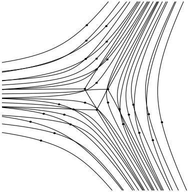
Fact

The BNR equation has two difference types of Stokes curves:

- Ordinary Stokes curve (Stokes curve emanating from ordinary turning point)
- New Stokes curve (Stokes curve emanating from virtual turning point)

In general, higher order ordinary differential equation has infinitely many virtual turning points and new Stokes curves





Remarks

- An ordinary turning point is a lagrange singularity for $\{p(x_1, \zeta_1) = 0\} \subset T^*\mathbb{C}$
- A (ordinary / virtual) turning point is a singularity of the bicharacteristic curve for the (micro)differential operator, for example,
$$\frac{\partial^3}{\partial x_1^3}\frac{\partial^{-3}}{\partial y^{-3}} + \frac{1}{2}c\frac{\partial}{\partial x_1}\frac{\partial^{-1}}{\partial y^{-1}} + \frac{1}{4}x_1$$

Exact WKB analysis for the Pearcey system (1/5)

The Pearcey system:

$$\begin{cases} \left(\eta^{-3}\dfrac{\partial^3}{\partial x_1^3} + \dfrac{x_2}{2}\eta^{-1}\dfrac{\partial}{\partial x_1} + \dfrac{x_1}{4}\right)\psi = 0, \\ \left(\eta^{-1}\dfrac{\partial}{\partial x_2} - \eta^{-2}\dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0 \end{cases}$$

Definition: WKB solution, turning point, Stokes surface

- WKB solution is given by the following formal power series

$$\psi_j = \exp\left(\eta\int^x \omega_j dx\right)\sum_{n=0}^{\infty}\eta^{-(n+1/2)}\psi_{j,n}(x)$$

where $\omega_j = \zeta_{1,j}(x)dx_1 + \zeta_{2,j}(x)dx_2$: closed 1-form, $\zeta_{1,j}$ and $\zeta_{2,j}$ satisfy

$$\begin{cases} p_1(x_1, x_2, \zeta_1) = \zeta_1^3 + \dfrac{1}{2}x_2\zeta_1 + \dfrac{1}{4}x_1 = 0, \\ p_2(\zeta_1, \zeta_2) = \zeta_2 - \zeta_1^2 = 0 \end{cases}$$

- A point $a \in \mathbb{C}^2$ is called a turning point if there exist two 1-forms ω_j and $\omega_{j'}$ such that $\omega_j(a) = \omega_{j'}(a)$, i.e., $\zeta_{1,j}(a) = \zeta_{1,j'}(a)$ & $\zeta_{2,j}(a) = \zeta_{2,j'}(a)$
- A Stokes surface emanating from $x = a$ is defined by

$$\Im\int_a^{x_1}(\omega_j - \omega_{j'}) = 0$$

Exact WKB analysis for the Pearcey system (2/5)

Fact

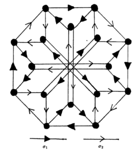
- The set of the turning points is given by $\{x \in \mathbb{C}^2 \ ; \ 27x_1^2 + 8x_2^3 = 0\}$
  In particular, this set has the cuspidial singularity at $x = 0$

Fact

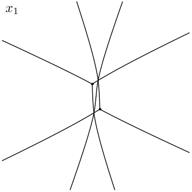
- The Stokes surface is expressed by the family of algebraic varieties

$$\bigcup_{c \in \mathbb{R}} \left\{ x \in \mathbb{C}^2 \mid F(x,w) = 0, \ \frac{F(x,w+c) - F(x,w)}{c} = 0 \text{ for some } w \in \mathbb{C} \right\}$$

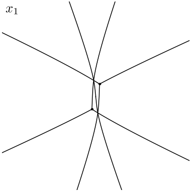
  where $F(x,w)$ is a discriminant of $t^4 + x_2 t^2 + x_1 t - w$ w.r.t the variable t
- The configuration graph of the Stokes geometry is the following graph (Jianming-Brieskorn, 1990)



The Stokes geometries for $x_2 = 1 \pm \varepsilon\sqrt{-1}$ $(0 < \varepsilon \ll 1)$ are given by the following figures



$x_2 = 1 + \varepsilon\sqrt{-1}$



$x_2 = 1 - \varepsilon\sqrt{-1}$

Main Result1

- Ordinary Stokes curves for the BNR equation are contained in the Stokes surface for the Pearcey system
- A new Stokes curve for the BNR equation is also contained in the Stokes surface for the Pearcey system

- These facts also hold for other examples
- In the proof of this fact, the cuspidal singularity of the set of turning points plays an important role
- We want to study the behavior of a holonomic system near such a cuspidal singularity

Exact WKB analysis for the Pearcey system (4/5)

Main Result2

Every (singular perturbed) holonomic system with two independent variables can be transformed to the Pearcey system at a cuspidal singularity of the set of the turning points

A completely integrable system:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x,\eta)\Psi, & P(x,\eta) = \displaystyle\sum_{m\geq 0}\eta^{-m}P_m(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, & Q(x,\eta) = \displaystyle\sum_{m\geq 0}\eta^{-m}Q_m(x) \end{cases}$$

where $P_m(x), Q_m(x)$ are 3×3 matrices with holomorphic entries near $x=0$

Suppose that "charactistic equation" $\det(\zeta_1 - P_0(x))$ is a versal unfolding of ζ_1^3 at $x=0$

Then the completely integrable system is transformed into the following system

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x)\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, \end{cases} \quad P(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix},\ Q(x,\eta) = \left(P_0^2 + \frac{x_2}{3}\right) + \eta^{-1}\frac{\partial P_0}{\partial x_1}$$

equivalent to the Pearcey system by a gauge transformation and a coordinate transformation

Exact WKB analysis for the Pearcey system (5/5)

Normal form of function:

- $F(x,\zeta_1)$: a versal unfolding of ζ_1^3
  $\Longrightarrow F(x,\zeta_1)$ is trasformed into the normal form $\zeta_1^3 + \dfrac{1}{2}x_2\zeta_1 + \dfrac{1}{4}x_1$

Normal form of completely integrable system:

- A completely integrable system

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x,\eta)\Psi, & P(x,\eta) = \displaystyle\sum_{m\geq 0}\eta^{-m}P_m(x), \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x,\eta)\Psi, & Q(x,\eta) = \displaystyle\sum_{m\geq 0}\eta^{-m}Q_m(x) \end{cases}$$

such that a characteristic equation of this system is a versal unfolding of ζ_1^3
$\Longrightarrow$ This completely integrable is trasformed into the Pearcey system

Relationship between the BNR equation & the Pearcey system (1/3)

The BNR equation

$$\left(\eta^{-3}\frac{d^3}{dx_1^3}+\frac{c}{2}\eta^{-1}\frac{d}{dx_1}+\frac{x_1}{4}\right)\psi=0$$

is given by the restricting the Pearcey system

$$\begin{cases}\left(\eta^{-3}\frac{\partial^3}{\partial x_1^3}+\frac{x_2}{2}\eta^{-1}\frac{\partial}{\partial x_1}+\frac{x_1}{4}\right)\psi=0,\\ \left(\eta^{-1}\frac{\partial}{\partial x_2}-\eta^{-2}\frac{\partial^2}{\partial x_1^2}\right)\psi=0\end{cases}$$

to $x_2=c$

Question:
What is a good (systematic) construction of the Pearcey system from the BNR equation?

Main Result3

We give an algorithm for the construction of the Pearcey system from the BNR equation

Remark

The following algorithm can be applied to other equations

Relationship between the BNR equation & the Pearcey system (2/3)

We first discuss the case of completely integrable Hamiltonian

1. We consider a completely integrable Hamiltonian:

$$\zeta_1^3 + \frac{c}{2}\zeta_1 + \frac{x_1}{4}$$

2. We add a trivial first integral ζ_2:

$$\begin{cases} \zeta_1^3 + \frac{c}{2}\zeta_1 + \frac{x_1}{4}, \\ \zeta_2 \end{cases}$$

3. We apply the canonical transformation defined by the generating function

$$x_1\widetilde{\zeta}_1 + x_2\widetilde{\zeta}_2 + x_2\widetilde{\zeta}_1^2$$

to this completely integrable Hamiltonian:

$$\begin{cases} \zeta_1^3 + \frac{c}{2}\zeta_1 + \frac{x_1 + 2x_2\zeta_1}{4} = \zeta_1^3 + \frac{x_2 + c}{2}\zeta_1 + \frac{x_1}{4}, \\ \zeta_2 - \zeta_1^2 \end{cases}$$

In what follows, we identify η and $\dfrac{\partial}{\partial y}$

1. We consider a ordinary differential equation:

$$\left(\frac{\partial^{-3}}{\partial x_1^{-3}}\frac{\partial^3}{\partial x_1^3}+\frac{c}{2}\frac{\partial^{-1}}{\partial x_1^{-1}}\frac{\partial}{\partial x_1}+\frac{x_1}{4}\right)\psi=0$$

2. We add a trivial differential equation:

$$\begin{cases}\left(\dfrac{\partial^{-3}}{\partial x_1^{-3}}\dfrac{\partial^3}{\partial x_1^3}+\dfrac{c}{2}\dfrac{\partial^{-1}}{\partial x_1^{-1}}\dfrac{\partial}{\partial x_1}+\dfrac{x_1}{4}\right)\psi=0,\\ \left(\dfrac{\partial^{-1}}{\partial y^{-1}}\dfrac{\partial}{\partial x_2}\right)\psi=0\end{cases}$$

This system is holonomic, if we consider that $\frac{\partial}{\partial y}$ is a parameter
3. We apply the quantized canonical transformation defined by the homogenieous canonical transformation given by the generating function

$$x_1\widetilde{\xi}_1+x_2\widetilde{\xi}_2+y\widetilde{\eta}+x_2\widetilde{\eta}^{-1}\widetilde{\xi}_1^2$$

to this system:

$$\begin{cases}\left(\dfrac{\partial^{-3}}{\partial x_1^{-3}}\dfrac{\partial^3}{\partial x_1^3}+\dfrac{c}{2}\dfrac{\partial^{-1}}{\partial x_1^{-1}}\dfrac{\partial}{\partial x_1}+\dfrac{1}{4}\left(x_1+2x_2\dfrac{\partial^{-1}}{\partial y^{-1}}\dfrac{\partial}{\partial x_1}\right)\right)\psi=0,\\ \dfrac{\partial^{-1}}{\partial y^{-1}}\left(\dfrac{\partial}{\partial x_2}-\dfrac{\partial^{-1}}{\partial y^{-1}}\dfrac{\partial^2}{\partial x_1^2}\right)\psi=0\end{cases}$$

Summary

We discuss the following three topics

- Ordinary Stokes curves and a new Stokes curve for the BNR equation are contained in the Stokes surface for the Pearcey system
- Every (singular perturbed) holonomic system with two independent variables can be transformed to the Pearcey system at a cuspidal singularity of the set of the turning points
- An algorithm for the construction of the Pearcey system from the BNR equation