k-summability of formal solutions for certain partial differential equations with time dependent polynomial coefficients

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Thursday, September 17, 2015

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Result

$$L = \partial_t - P(t, \partial_x), \quad P(t, \partial_x) = \sum_{\alpha \in \mathbb{N}_0}^{\text{finite}} a_\alpha(t) \partial_x^\alpha,$$

where $t, x \in \mathbb{C}$, $a_\alpha(t) \in \mathbb{C}[t]$ $(\forall \alpha)$ and $\mathbb{N}_0 := \{0, 1, \cdots\}$.

We consider the following Cauchy problem

$$\begin{cases} LU(t,x) = (\partial_t - P(t, \partial_x))U(t,x) = 0, \\ U(0,x) = \varphi(x) \in \mathcal{O}_x. \end{cases} \tag{CP}$$

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$\exists 1$ formal solution :

$$\hat{U}(t,x) = \sum_{n \geq 0} U_n(x) \frac{t^n}{n!}, \quad U_0(x) = \varphi(x).$$

$$P(t,\partial_x) = \sum_{\alpha\in\mathbb{N}_0}^{\text{finite}} a_\alpha(t)\partial_x^\alpha$$

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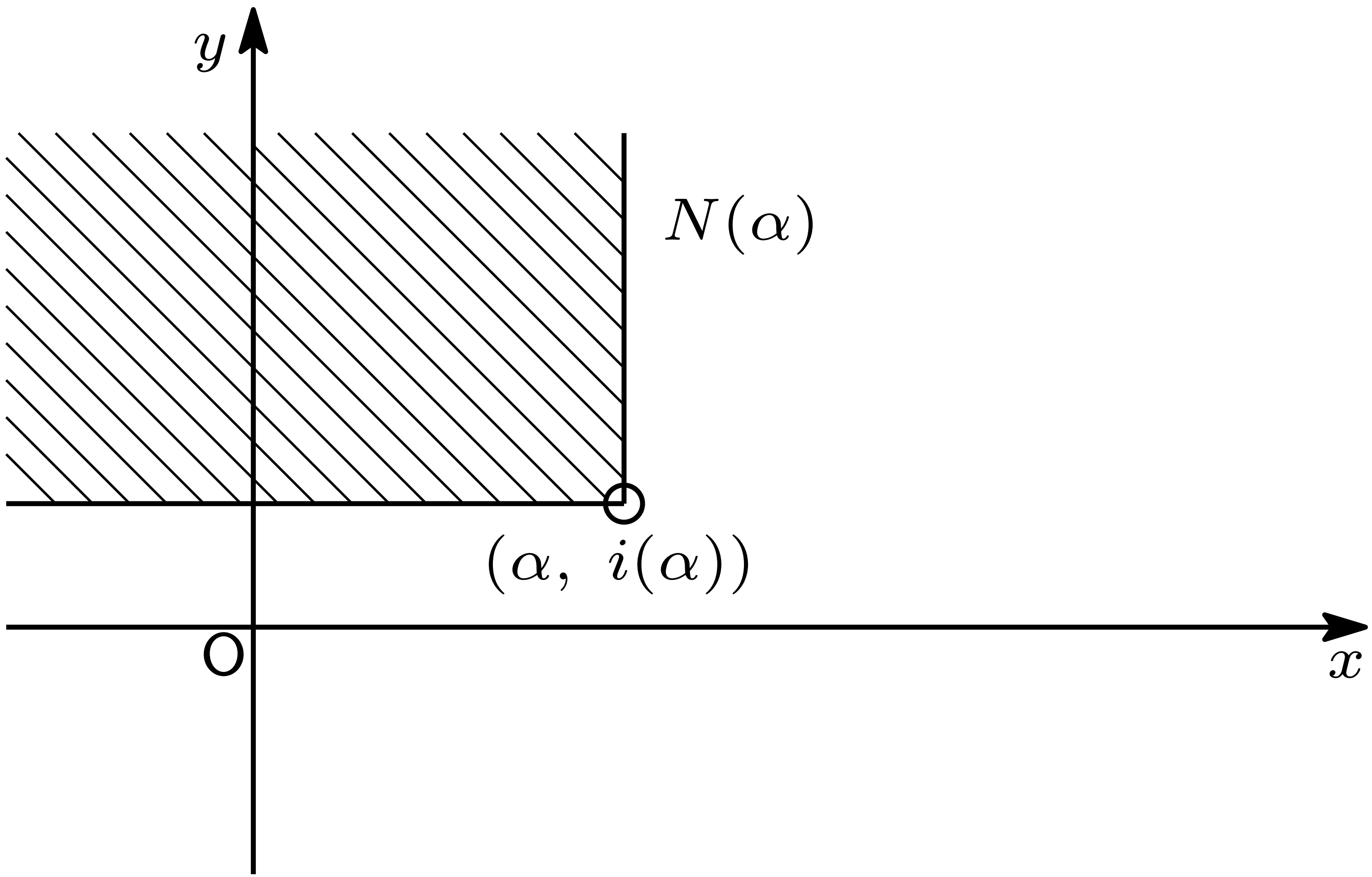
Assumption 1 (A-1) [non-Kowalevskian]

$$\alpha_* := \max\{\alpha;\ a_\alpha(t) \not\equiv 0\} \geq 2.$$

Aim: k-summability of $\hat{U}(t,x)$ under some conditions for L.

Newton polygon $i(\alpha)$: the order of zero of $a_\alpha(t)$ at $t = 0$.

$$\mathrm{N}(\alpha) := \begin{cases} \{(x, y) \in \mathbb{R}^2; x \le \alpha,\ y \ge i(\alpha), a_\alpha(t) \not\equiv 0\}, \\ \phi \qquad \text{for } a_\alpha(t) \equiv 0. \end{cases}$$



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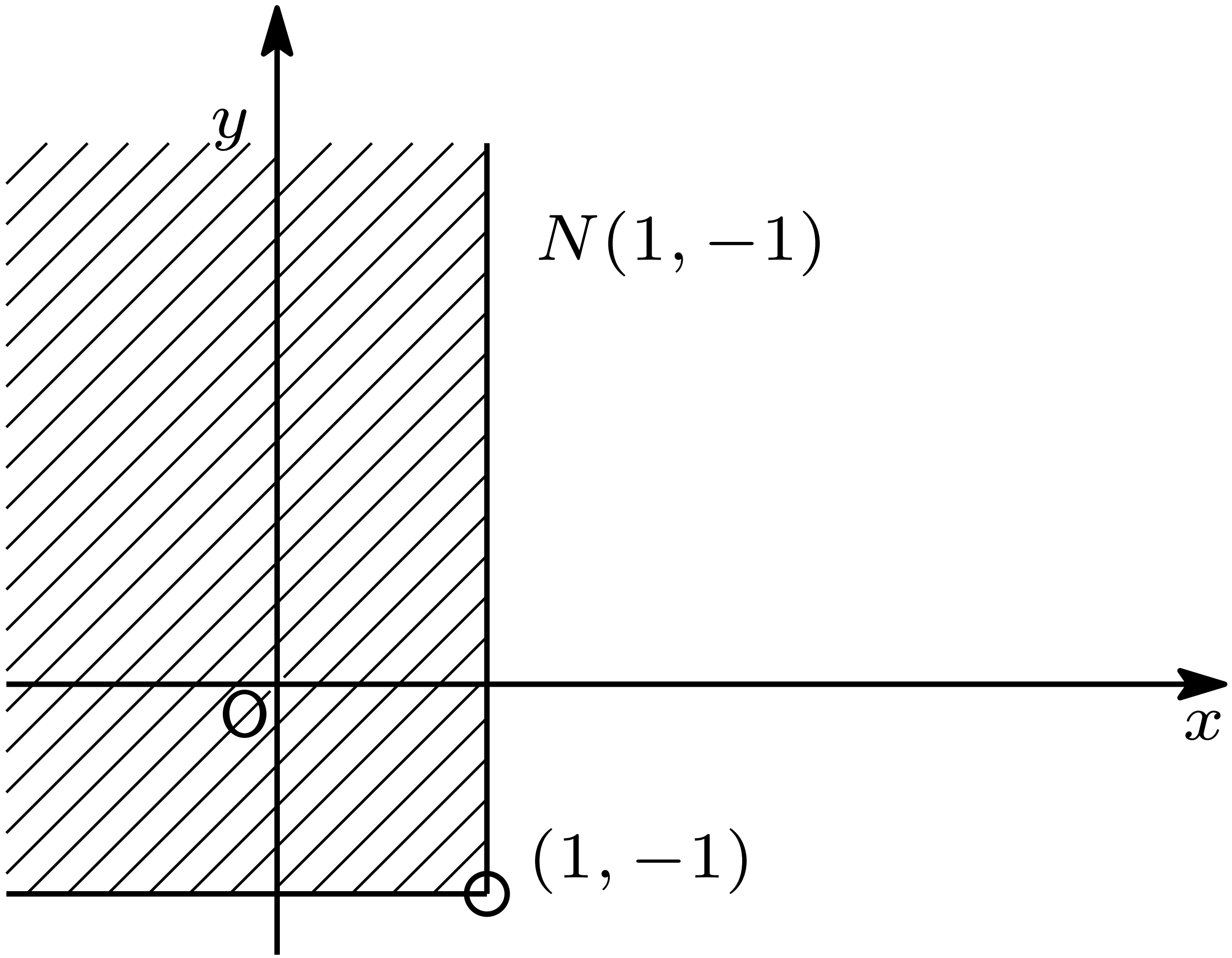
and $\mathrm{N}(1, -1) := \{(x, y) \in \mathbb{R}^2; x \leq 1, y \geq -1\}$.



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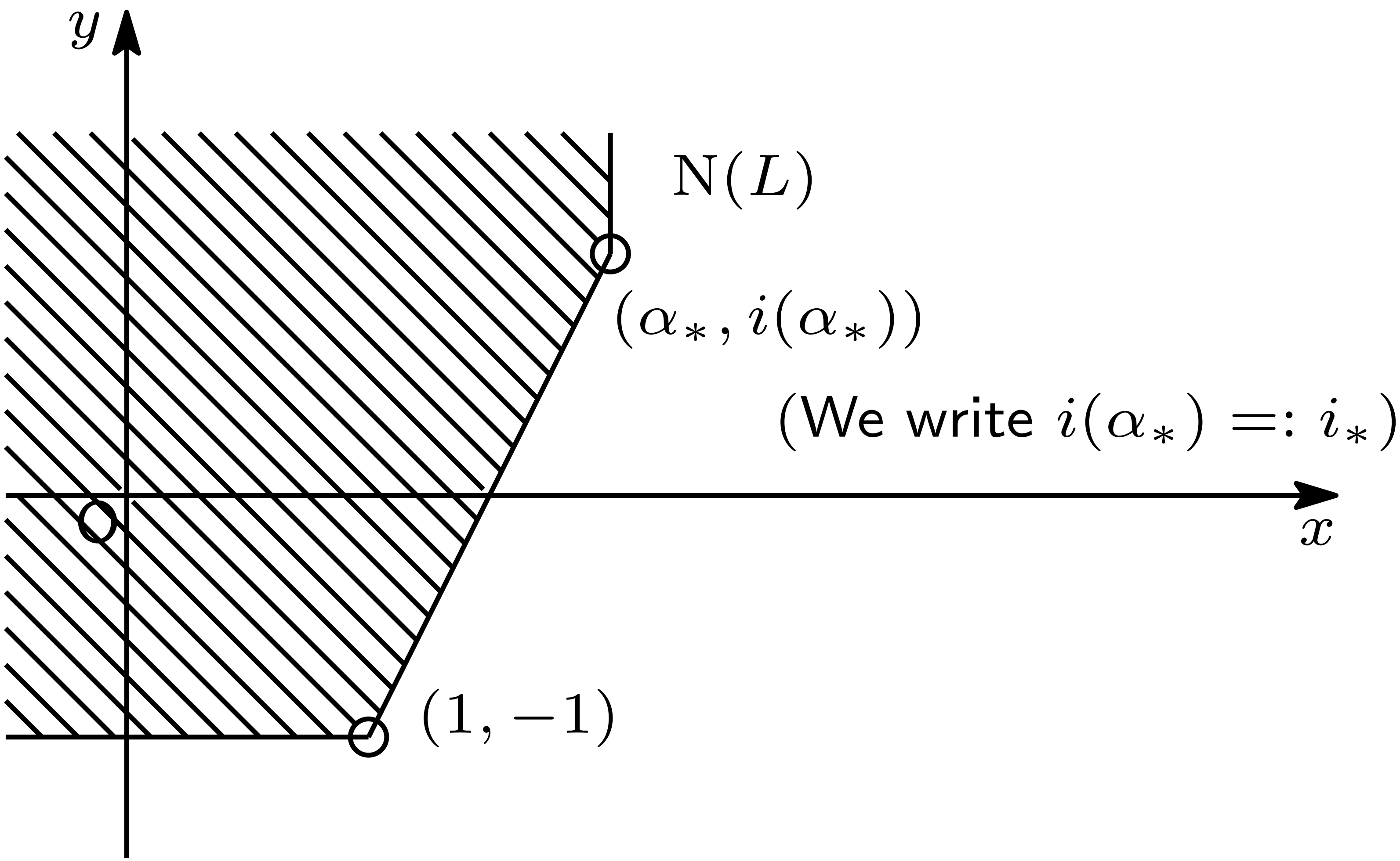
$$\mathrm{N}(L) := \mathrm{Ch}\left\{\mathrm{N}(-1, 1) \cup \bigcup_{\alpha \in \mathbb{N}_0} \mathrm{N}(\alpha)\right\}.$$

Assumption 2 (A-2) [Newton polygon] $\mathrm{N}(L)$ has only one side of a positive slope with $(1, -1)$ and $(\alpha_*, i(\alpha_*))$.



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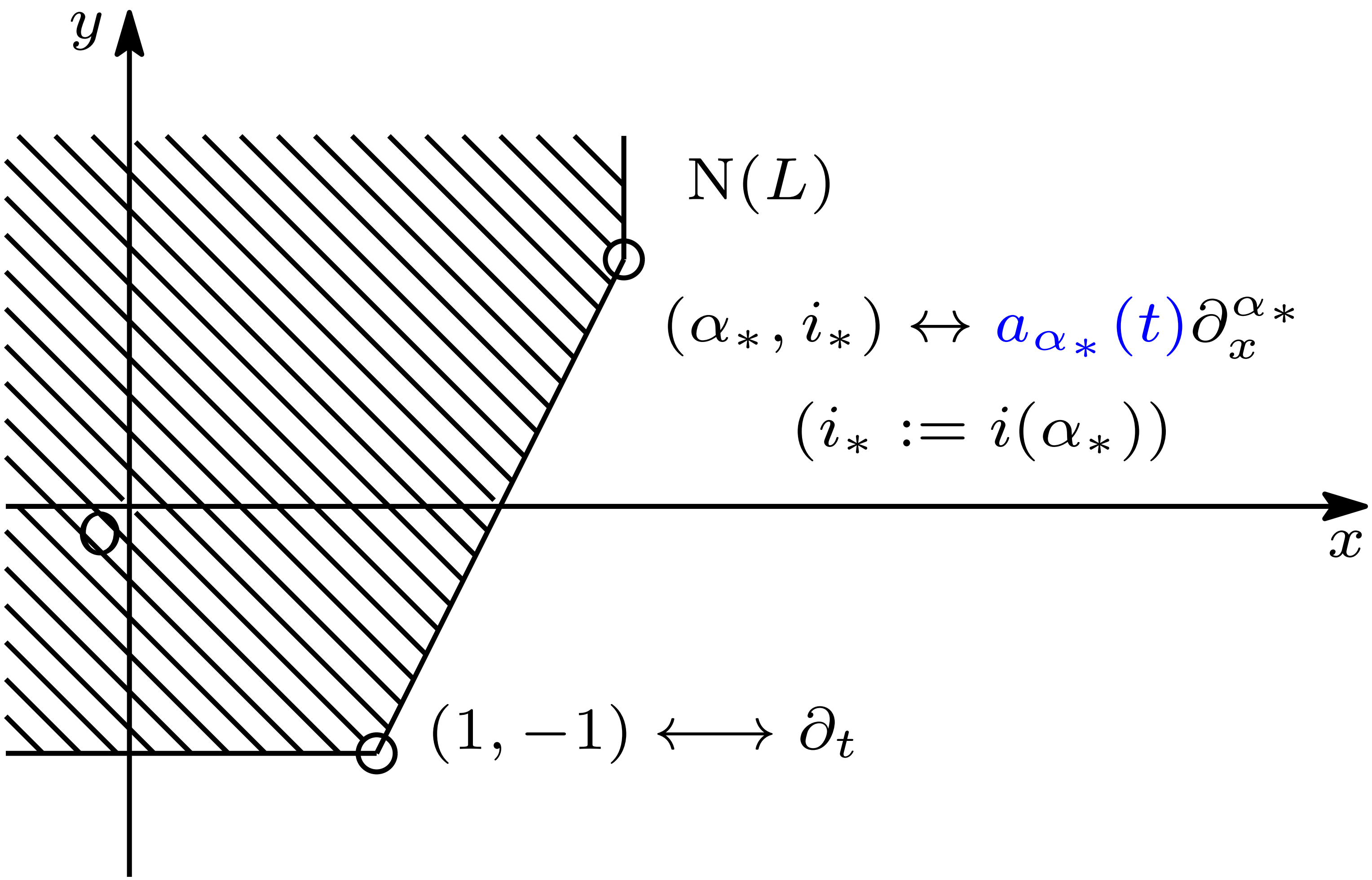
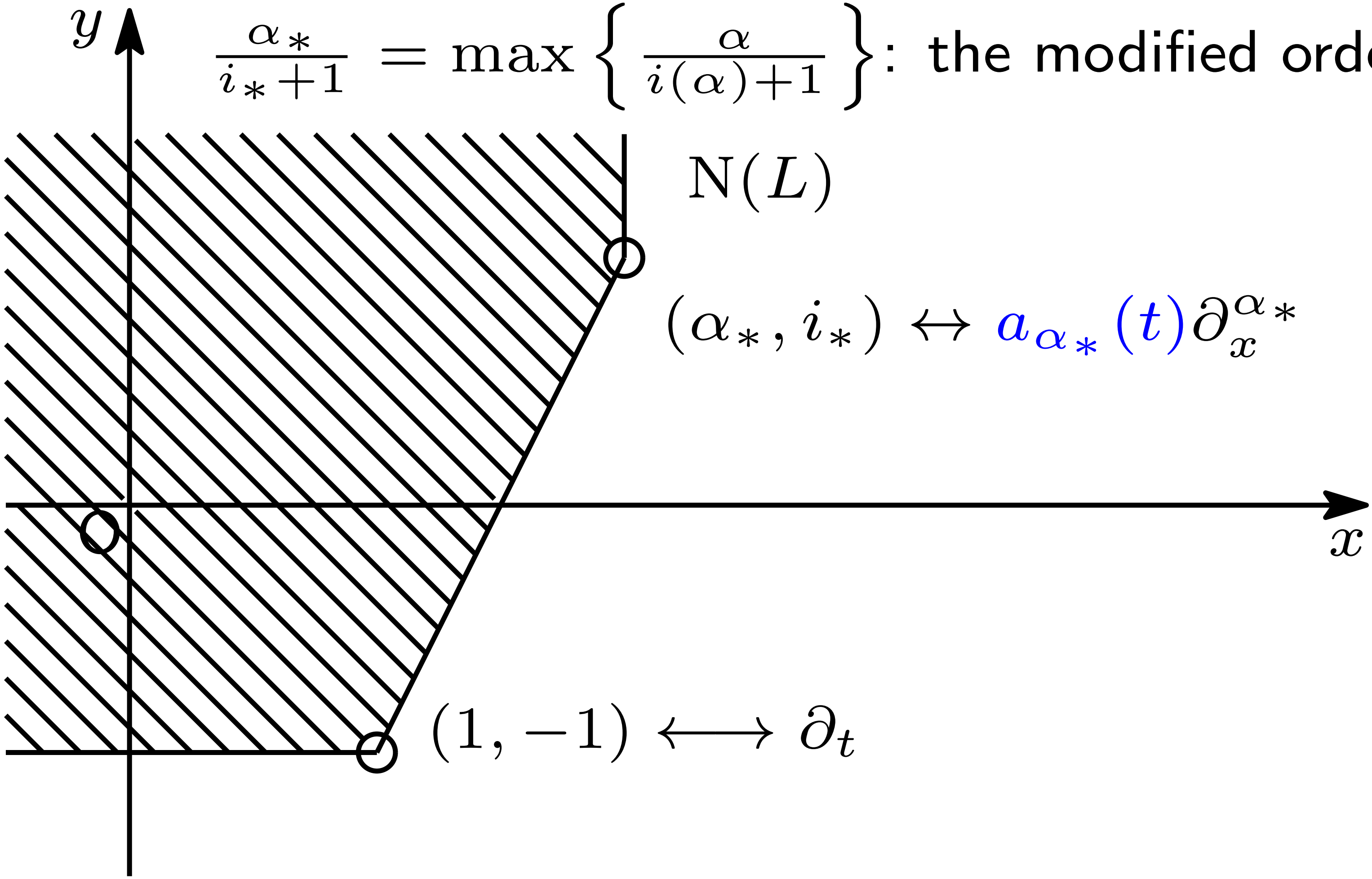


Assumption 3 (A-3) [modified order (M.O.)] For $P(t, \partial_x)$,

$$\frac{\alpha_*}{i_* + 1} \geq \frac{\alpha}{i(\alpha) + 1} \quad \text{for } \forall\alpha \text{ with } a_\alpha(t) \not\equiv 0.$$

$\frac{\alpha}{i(\alpha)+1}$: the modified order (m.o.) for $a_\alpha(t)\partial_x^\alpha$

$\frac{\alpha_*}{i_*+1} = \max\left\{\frac{\alpha}{i(\alpha)+1}\right\}$: the modified order of L.



Assumption 3 (A-3) [modified order] For $P(t, \partial_x)$,

$$\frac{\alpha_*}{i_* + 1} \geq \frac{\alpha}{i(\alpha) + 1} \quad \text{for } \forall\alpha \text{ with } a_\alpha(t) \not\equiv 0.$$

$\frac{\alpha_*}{i_*+1}$: the modified order of L and $\frac{\alpha_*}{i_*+1} = \frac{p}{q}$, $(p, q) = 1$.

Assumption 4 (A-4) [Technical condition]

$$a_\alpha(t) = \sum_{\substack{0 \leq i \leq i_* \\ i+1 \in q\mathbb{N}}} a_i^{(\alpha)} t^i.$$

Especially,

$$a_{\alpha_*}(t) = a_{i_*}^{(\alpha_*)} t^{i_*}.$$

Theorem.

Let $k := \dfrac{i_* + 1}{\alpha_* - 1}$. For a fixed $d \in \mathbb{R}$, we define

$d_n := \dfrac{q}{p} d + \dfrac{\arg a_{i_*}^{(\alpha_*)} + 2\pi n}{\alpha_*}$ for $n = 0, 1, \ldots, \alpha_* - 1$.

$$\varphi(x) \in \mathcal{O}\left(D_\rho \cup \bigcup_{n=0}^{\alpha_* - 1} S(d_j, \varepsilon) \right),$$

$$|\varphi(x)| \le C \exp\left(\delta |x|^{\frac{\alpha_*}{\alpha_* - 1}} \right). \quad (1)$$

Then under (A-1)-(A-4), $\hat{U}(t, x)$ of (CP) is k-summable in d direction. (We write "$\hat{U}(t, x) \in \mathcal{O}_x\{t\}_{k,d}$").

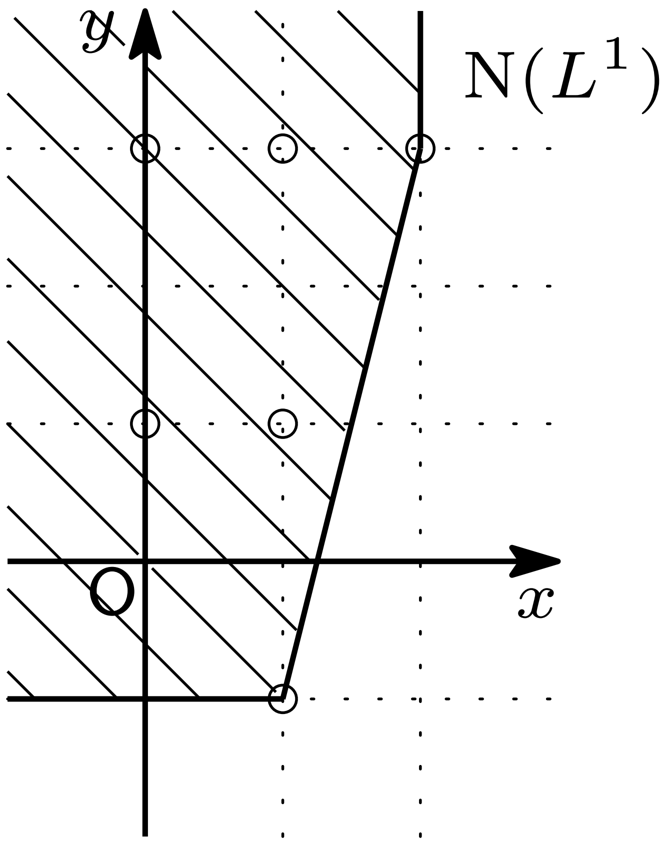
Here $D_\rho = \{x; |x| \le \rho\}$ and $S(d, \varepsilon) = \{x; |d - \arg x| < \frac{\varepsilon}{2}\}$.

Example

Example 1 $L^1 = \partial_t - \{t^3\partial_x^2 + (t^3+t)\partial_x + (t^3+t)\}$

$$=: \partial_t - P^1(t, \partial_x)$$

$(\alpha_*, i_*) = (2, 3)$, M.O.$=1/2$



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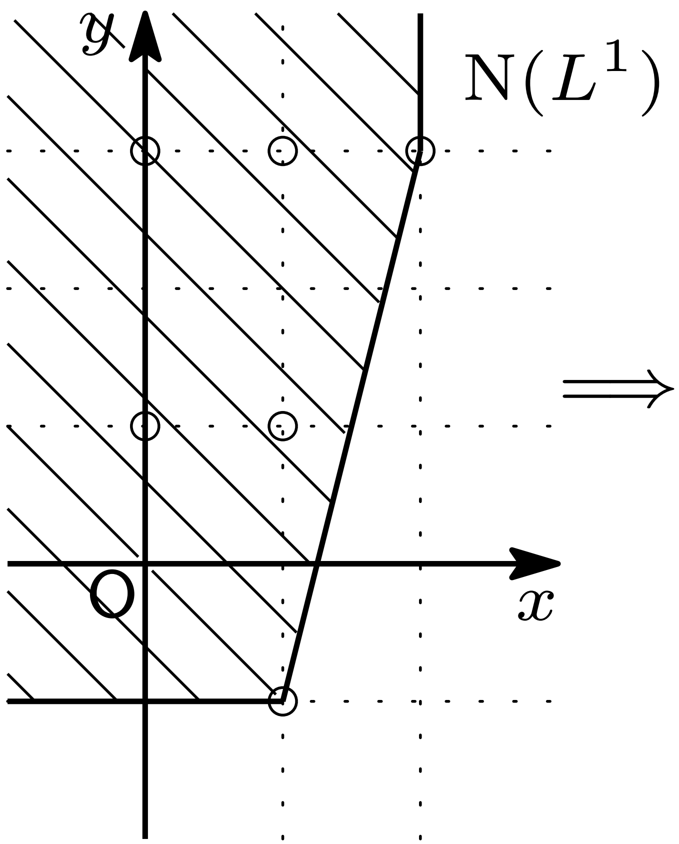
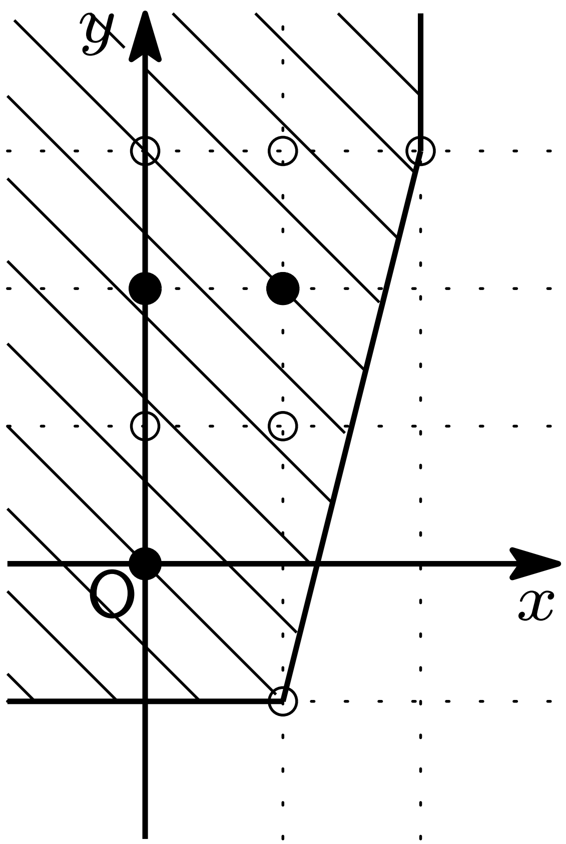


$\Longrightarrow$



Outline of proof of Theorem

- Construction of formal solution $\hat{U}(t,x)$ of (CP).

$$\hat{U}(t,x) = \sum_{\nu \geq 0} \hat{u}_\nu(t,x) \quad \hat{u}_\nu(t,x) : \text{successive approximation sol.}$$

- $\hat{U}(t,x) \in \mathcal{O}_x\{t\}_{k,d} \iff \hat{u}_\nu(t,x) \in \mathcal{O}_x\{t\}_{k,d}$.

Construction of $\hat{U}(t,x)$

We give a decomposition of $P(t,\partial_x)$.

For $\ell \geq 0$, $K_\ell := \left\{(\alpha,i); \ell = p(i+1)/q - \alpha, a_i^{(\alpha)} \neq 0\right\}$,

$P_\ell(t,\partial_x) := \sum_{K_\ell} a_i^{(\alpha)} t^i \partial_x^\alpha$. Then

$$P(t,\partial_x) = \sum_{\ell=0}^{\alpha_*} P_\ell(t,\partial_x).$$

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Example 1

$$\begin{aligned} P^1(t,x) &= t^3\partial_x^2 + (t^3+t)\partial_x + (t^3+t) \\ &= (t^3\partial_x^2 + t\partial_x) + (t^3\partial_x + t) + t^3 \\ &=: \qquad P_0 \qquad + \qquad P_1 \qquad + P_2 \end{aligned}$$

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We consider the Cauchy problems for each ν

$$\begin{cases} \partial_t u_\nu(t,x) = \displaystyle\sum_{\ell=0}^{\min\{\alpha_*,\nu\}} P_\ell(t,\partial_x) u_{\nu-\ell}(t,x), \\ u_\nu(0,x) = \varphi(x) \text{ if } \nu = 0, \quad 0 \text{ if } \nu \geq 1. \end{cases}$$

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$\Rightarrow$ $\hat{U}(t,x) = \displaystyle\sum_{\nu\geq 0} \hat{u}_\nu(t,x)$.

k-summability of $\hat{u}_\nu(t,x)$

$$\hat{u}_\nu(t,x) = \sum_{n\geq 0} A_\nu(n)\varphi^{(pn/q-\nu)}(x)\frac{t^n}{n!}$$

$$= \frac{1}{2\pi i}\oint \varphi(x+\zeta)\zeta^{\nu-1}\sum_{n\geq 0} A_\nu(n)\frac{(pn/q-\nu)!}{n!}\left(\frac{t}{\zeta^{p/q}}\right)^n d\zeta$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\tau^{-\nu}(1-\tau)^{\nu-1}}{(\nu-1)!}d\tau \oint \varphi(x+\zeta)\zeta^{\nu-1}\hat{g}_\nu\left(\frac{\tau^{p/q}t}{\zeta^{p/q}}\right)d\zeta,$$

where $\hat{g}_\nu(X) = \sum_{n\geq 0} A_\nu(n)\frac{\left(\frac{p}{q}n\right)!}{n!}X^n.$

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