Strongly regular sequences, proximate orders and summability

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Strongly regular sequences (following V. Thilliez)

$\mathbb{N}_0 = \{0, 1, 2, ...\}$

A sequence $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ of positive real numbers, with $M_0 = 1$, is said to be strongly regular if it is:

- logarithmically convex: $M_p^2 \leq M_{p-1}M_{p+1}, \quad p \geq 1.$
- of moderate growth: there exists a constant $A > 0$ such that

$$M_{l+p} \leq A^{l+p} M_l M_p, \quad l, p \in \mathbb{N}_0.$$

- strongly non-quasianalytic: there exists $B > 0$ such that

$$\sum_{k\geq p} \frac{M_k}{(k+1)M_{k+1}} \leq B\frac{M_p}{M_{p+1}}, \quad p \in \mathbb{N}_0.$$

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The sequence of quotients, $\boldsymbol{m} = (m_p := M_{p+1}/M_p)_{p \in \mathbb{N}_0}$, is an increasing sequence tending to infinity.

Examples of strongly regular sequences

If $\mathbb{M}$ is strongly regular, then $\mathbb{M}' = (p!M_p)_{p\in\mathbb{N}_0}$ verifies (M.1)+(M.2)+(M.3) of H. Komatsu.

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Examples:

- $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$.
- $\mathbb{M}_{\alpha,\beta} = \left(p!^\alpha \prod_{m=0}^{p} \log^\beta(e+m)\right)_{p\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.
- For $q > 1$, $\mathbb{M} = (q^{p^2})_{p\in\mathbb{N}_0}$ is logarithmically convex and strongly non-quasianalytic, but not of moderate growth.

$\mathbb{M}$-asymptotic expansion

We say $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ if f admits the series $\hat{f} = \sum_{p=0}^{\infty} a_p z^p$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f \sim_{\mathbb{M}} \hat{f}$: For every bounded proper subsector T of the sector S there exist $C_T, B_T > 0$ such that for every $z \in T$ and every $p \in \mathbb{N}_0$, we have

$$\Big| f(z) - \sum_{k=0}^{p-1} a_k z^k \Big| \leq C_T B_T^p M_p |z|^p.$$

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For every bounded proper subsector T of the sector S we check that $a_p = \lim_{\substack{z \to 0 \\ z \in T}} \dfrac{f^{(p)}(z)}{p!}$ and we can define $f^{(p)}(0) := a_p p!$

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$$\Lambda_{\mathbb{M}} = \Big\{ \boldsymbol{\mu} = (\mu_p)_{p \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : \exists A > 0 \text{ s.t. } |\boldsymbol{\mu}|_{\mathbb{M},A} := \sup_{p \in \mathbb{N}_0} \frac{|\mu_p|}{A^p p! M_p} < \infty \Big\}.$$

Injectivity of the Borel map

We consider the Borel map which is a homomorphism of differential algebras.

$$\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S) \longrightarrow \Lambda_{\mathbb{M}}$$
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$f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ is said to be flat if $\tilde{\mathcal{B}}(f)$ is the null sequence.

The class $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ is said to be quasianalytic if it does not contain nontrivial flat functions (in other words, the asymptotic Borel map is injective in this class).

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Notation: S_γ will denote the unbounded sector of opening $\pi\gamma$, bisected by direction 0.

Quasianalyticity is kept when enlarging the opening of the sector.We look for $\omega(\mathbb{M}) := \inf\{\gamma > 0 : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \text{ is quasianalytic}\}$.

Optimal opening in $\tilde{A}_{\mathbb{M}}(S_\gamma)$

Thanks to a theorem of B. I. Korenbljum we can compute $\omega(\mathbb{M})$.

B. I. Korenbljum, Conditions of nontriviality of certain classes of functions analytic in a sector, and problems of quasianalyticity, Soviet Math. Dokl. 7 (1966), 232–236.

Proposition (J. Sanz (2014))

For $\mathbb{M}$ *is strongly regular, we have* $\omega(\mathbb{M}) = \liminf_{p\to\infty} \dfrac{\log(m_p)}{\log(p)}$, *and* $\omega(\mathbb{M}) \in (0,\infty)$.

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Theorem (generalized Watson's lemma, partial version, J. Sanz (2014))

If the opening of S is larger (respectively, smaller) than $\pi\omega(\mathbb{M})$, then $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S) \longrightarrow \Lambda_{\mathbb{M}}$ is (resp. is not) injective.

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Aim: to construct nontrivial flat functions in optimal sectors.

Null asymptotics

Theorem (Thilliez)

The following are equivalent: (i) $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ *and* $f \sim_{\mathbb{M}} \hat{0}$.
(ii) For every bounded proper subsector T *of* S *there exist* $c_1, c_2 > 0$ *with*

$$|f(z)| \le c_1 e^{-M(1/(c_2|z|))}, \qquad z \in T,$$

where $M(t) := \sup_{p \in \mathbb{N}_0} \log\big(\frac{t^p}{M_p}\big)$ *for* $t > 0$ *and* $M(0) = 0$.

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If we find a holomorphic function $V(z)$ in $S_{\omega(\mathbb{M})}$ whose growth is suitably governed by the function $M(t)$, then $\exp(-V(1/z))$ will be flat.

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Example: Gevrey of order $1/k > 0$, i.e. $\mathbb{M}_{1/k} = (p!^{1/k})_{p \in \mathbb{N}_0}$; we write $\tilde{\mathcal{A}}_{1/k}(S)$.
For large t, $c_2 t^k \le M_{1/k}(t) \le c_1 t^k \quad \Rightarrow$

$$\text{Flatness} \;\equiv\; \text{Exponential decrease of order } k.$$

Putting $V(z) = z^k$, $\exp(-z^{-k})$ is flat in $\tilde{\mathcal{A}}_{1/k}(S_{1/k})$.

Proximate orders

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Definition (E. Lindelöf, G. Valiron)

We say $\rho(t) : (a, \infty) \to \mathbb{R}$ is a proximate order if the following hold:

(1) ρ is continuous and piecewise continuously differentiable,

(2) $\rho(t) \geq 0$ for every $r > a > 0$,

(3) $\lim_{t\to\infty} \rho(t) = \rho < \infty$,

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Examples:

- Positive constant $\rho(t) = \rho \geq 0$.
- Given $\rho > 0$, $\rho(t) = \rho + \frac{1}{\log(t)}$.
- In $(1, \infty)$, the function $\rho(t) = 1 + \frac{\sin(t)}{t}$ does not verify (4).

Motivation of the problem

Given $\mathbb{M}$ we look for $d_{\mathbb{M}}(t) := \log(M(t))/\log(t)$.

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Under the assumption that $d_{\mathbb{M}}(t)$ is a proximate order and using the work of L. S. Maergoiz, J. Sanz (2014) gave this result.

L. S. Maergoiz, Indicator diagram and generalized Borel-Laplace transforms for entire functions of a given proximate order, St. Petersburg Math. J. 12 (2001), 191–232.

Theorem (generalized Watson's lemma, J. Sanz (2014))

Suppose $d_{\mathbb{M}}(t)$ is a proximate order. Then, $\tilde{A}_{\mathbb{M}}(S)$ is quasianalytic if, and only if, the opening of S is greater than $\pi\omega(\mathbb{M})$.

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Aim: Characterize when $d_{\mathbb{M}}(t)$ is a proximate order.

Main result

We want to know when $d_{\mathbb{M}}(t)$ verifies conditions (3) ($\lim_{t\to\infty} d_{\mathbb{M}}(t) < \infty$) and (4) ($\lim_{t\to\infty} t{d'}_{\mathbb{M}}(t)\log(t) = 0$) of proximate orders.

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We always have that $\limsup_{t\to\infty} d_{\mathbb{M}}(t) = \omega(\mathbb{M})^{-1}$ and $\liminf_{p\to\infty} \dfrac{\log(m_p)}{\log(p)} = \omega(\mathbb{M})$.

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Theorem (J. J.-G, J. Sanz(2015))

Let $\mathbb{M}$ be a strongly regular sequence, the following statements are equivalent:

(a) $d_{\mathbb{M}}(t)$ *is a proximate order, i.e., verifies (3) and (4).*

(b) $\lim_{t\to\infty} d_{\mathbb{M}}(t) = \lim_{t\to\infty} \dfrac{\log(M(t))}{\log(t)} = \dfrac{1}{\omega(\mathbb{M})}$, *i.e,* $d_{\mathbb{M}}(t)$ *verifies (3).*

(c) $\lim_{p\to\infty} \dfrac{\log(m_p)}{\log(p)} = \omega(\mathbb{M})$.

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(c) $\lim_{p\to\infty} \dfrac{\log(m_p)}{\log(p)} = \omega(\mathbb{M})$.

We observe that, given $\mathbb{M}$, condition (c) is easy to check. Indeed, all the examples we know verify this assumption, but we do not know if this is always the case!

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Definition

Let $\mathbb{M}$ and $\mathbb{L}$ be a strongly regular sequences, we say they are equivalent if there exists $a > 0$ such that $a^{-1} l_p \leq m_p \leq a l_p$ for every $p \in \mathbb{N}$, where $(m_p)_p$ and $(\ell_p)_p$ are the sequence of quotients associated to $\mathbb{M}$ and $\mathbb{L}$.

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Lemma (J. J.-G., J. Sanz(2015))

Let $\mathbb{M}$ be a strongly regular sequence, then there exists an equivalent strongly regular sequence $\mathbb{L}$ such that $\liminf_{p\to\infty} \log\left(\frac{\ell_p}{L_p^{1/p}}\right) > 0.$

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Let $\mathbb{M}$ and $\mathbb{L}$ be a strongly regular sequences, we say they are equivalent if there exists $a > 0$ such that $a^{-1} l_p \le m_p \le a l_p$ for every $p \in \mathbb{N}$, where $(m_p)_p$ and $(\ell_p)_p$ are the sequence of quotients associated to $\mathbb{M}$ and $\mathbb{L}$.

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Let $\mathbb{M}$ be a strongly regular sequence, then there exists an equivalent strongly regular sequence $\mathbb{L}$ such that $\liminf_{p\to\infty} \log\left(\frac{\ell_p}{L_p^{1/p}}\right) > 0.$

H.-J. Petzsche, On E. Borel's theorem, Math. Ann. 282 (1988), no. 2, 299–313.

Since $\mathbb{L}$ is equivalent to $\mathbb{M}$, $\mathbb{L}$ satisfies (b). By this Lemma and (b) we see that $\mathbb{L}$ verifies (c). As condition (c) is stable under equivalence it also holds for $\mathbb{M}$.

Idea of the proof II

(c) $(\lim_{p\to\infty} \log(m_p)/\log(p) = \omega(\mathbb{M})) \Rightarrow$ (b) $(\lim_{t\to\infty} d_{\mathbb{M}}(t) = \omega(\mathbb{M})^{-1})$

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There exists an strongly regular sequence $\mathbb{L}$ equivalent to $\mathbb{M}$ and $h_1, h_2 > 0$ such that $\log((p-1)\log(h_1))/\log(l_p) \leq d_{\mathbb{L}}(t) \leq \log(p\log(h_2))/\log(l_{p-1})$ for every $t \in [l_{p-1}, l_p)$.

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As (c) holds for $\mathbb{M}$, $\mathbb{L}$ also satisfies it.

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As (c) holds for $\mathbb{M}$, $\mathbb{L}$ also satisfies it.Taking limit we deduce that $\mathbb{L}$ verifies (b) that is also stable under equivalence. Consequently, $\mathbb{M}$ verifies (b).

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(c) $(\lim_{p\to\infty} \log(m_p)/\log(p) = \omega(\mathbb{M})) \Rightarrow$ (b) $(\lim_{t\to\infty} d_{\mathbb{M}}(t) = \omega(\mathbb{M})^{-1})$

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We will show (b)+(c) $\Rightarrow$ (a) (condition (4)).

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Proposition (J. Sanz (2014))

If $d_{\mathbb{M}}(t)$ verifies (b) then $d_{\mathbb{M}}(t)$ is a proximate order (verifies (4)) if, and only if,

$$\lim_{p\to\infty} \log\left(\frac{m_p}{M_p^{1/p}}\right) = \omega(\mathbb{M}) \qquad \beta_p := \log\left(\frac{m_p}{M_p^{1/p}}\right).$$

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As (c) holds for $\mathbb{M}$, $\mathbb{L}$ also satisfies it. Taking limit we deduce that $\mathbb{L}$ verifies (b) that is also stable under equivalence. Consequently, $\mathbb{M}$ verifies (b).

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Let us see that (c) implies $\lim_{p\to\infty} \beta_p = \omega(\mathbb{M})$.

Idea of the proof III

Lemma (J. J.-G., J. Sanz(2015))

We have that $\dfrac{\log(m_p)}{\log p} = \dfrac{1}{\log p}\displaystyle\sum_{k=0}^{p-1}\frac{\beta_k}{k+1} + \frac{\beta_p}{\log p}.$

Idea of the proof III

Lemma (J. J.-G., J. Sanz(2015))

We have that $\dfrac{\log(m_p)}{\log p} = \dfrac{1}{\log p}\displaystyle\sum_{k=0}^{p-1}\frac{\beta_k}{k+1} + \frac{\beta_p}{\log p}.$

Since $\lim_{p\to\infty} \log(m_p)/\log(p) = \omega(\mathbb{M})$ and β_p is bounded we deduce that

$$\lim_{p\to\infty} \frac{1}{\log p}\sum_{k=0}^{p-1}\frac{\beta_k}{k+1} = \omega(\mathbb{M}).$$

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Definition

A numerical sequence $(s_k)_{k\in\mathbb{N}}$ of complex numbers is said to be Riesz summable, if there exists some $A \in \mathbb{C}$ such that $\lim_{p\to\infty}\dfrac{1}{\log p}\displaystyle\sum_{k=1}^{p}\frac{s_k}{k} = A$.

Idea of the proof III

Lemma (J. J.-G., J. Sanz(2015))

We have that $\dfrac{\log(m_p)}{\log p} = \dfrac{1}{\log p}\sum_{k=0}^{p-1}\dfrac{\beta_k}{k+1} + \dfrac{\beta_p}{\log p}$.

Since $\lim_{p\to\infty} \log(m_p)/\log(p) = \omega(\mathbb{M})$ and β_p is bounded we deduce that

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Definition

A numerical sequence $(s_k)_{k\in\mathbb{N}}$ of complex numbers is said to be Riesz summable, if there exists some $A\in\mathbb{C}$ such that $\lim_{p\to\infty}\dfrac{1}{\log p}\sum_{k=1}^{p}\dfrac{s_k}{k} = A$.

The sequence $(\beta_p)_{p\in\mathbb{N}_0}$ is Riesz summable to $\omega(\mathbb{M})$.

Idea of the proof IV

Theorem (F. Moricz (2013))

If a sequence $(s_k)_{k\in\mathbb{N}}$ *of real numbers is Riesz summable, then* $\lim_{k\to\infty} s_k$ *exists, if and only if,*

$$\limsup_{\lambda\to 1^+} \liminf_{p\to\infty} \frac{1}{(\lfloor p^\lambda \rfloor - p)H_p} \sum_{k=p+1}^{\lfloor p^\lambda \rfloor} \frac{s_k - s_p}{k} \geq 0 \text{ and}$$

$$\limsup_{\lambda\to 1^-} \liminf_{p\to\infty} \frac{1}{(p - \lfloor p^\lambda \rfloor)H_p} \sum_{k=\lfloor p^\lambda \rfloor+1}^{p} \frac{s_p - s_k}{k} \geq 0.$$

F. Moricz, Necessary and sufficient Tauberian conditions for the logarithmic summability of functions and sequences, Studia Math. 219 (2013), no. 2, 109–121.

Idea of the proof IV

Theorem (F. Moricz (2013))

If a sequence $(s_k)_{k\in\mathbb{N}}$ of real numbers is Riesz summable, then $\lim_{k\to\infty} s_k$ exists, if and only if,

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$(\beta_p)_{p\in\mathbb{N}_0}$ verifies these conditions, and consequently $\lim_{p\to\infty} \beta_p = \omega(\mathbb{M})$.

Thank you for your attention

References

J. Jiménez-Garrido, J. Sanz, Strongly regular sequences, proximate orders and kernels of summability, preprint, in preparation.

H. Komatsu, Ultradistributions, I: Structure theorems and a characterization, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 25–105.

B. I. Korenbljum, Conditions of nontriviality of certain classes of functions analytic in a sector, and problems of quasianalyticity, Soviet Math. Dokl. 7 (1966), 232–236.

L. S. Maergoiz, Indicator diagram and generalized Borel-Laplace transforms for entire functions of a given proximate order, St. Petersburg Math. J. 12 (2001), no. 2, 191–232.

F. Moricz, Necessary and sufficient Tauberian conditions for the logarithmic summability of functions and sequences, Studia Math. 219 (2013), no. 2, 109–121.

H.-J. Petzsche, On E. Borel's theorem, Math. Ann. 282 (1988), no. 2, 299–313.

J. Sanz, Flat functions in Carleman ultraholomorphic classes via proximate orders, J. Math. Anal. Appl. 415 (2014), 623–643.

V. Thilliez, Division by flat ultradifferentiable functions and sectorial extensions, Results Math. 44 (2003), 169–188.