Orthogonal polynomials and General Schlesinger systems

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- $w(t)$: weight fct. on $I \subset \mathbb{R}$
- $(f, g) = \int_I f(t)g(t)w(t)dt$: inner product.
- $p_n(t) = t^n + \cdots$ monic orthogonal poly. $n = 0, 1, \ldots$.
- $(p_m, p_n) = \delta_{nm} h_n$, where $h_n = (p_n, p_n)$.

Introduction

Classical orthogonal polynomials

- $w(t) = t^\alpha(1-t)^\beta$, $\rightsquigarrow$ Jacobi polynomial
- $w(t) = t^\alpha e^{-t}$, $\rightsquigarrow$ Laguerre polynomial
- $w(t) = e^{-t^2}$, $\rightsquigarrow$ Hermite polynomial

These weight functions are integrands of

$$B(\alpha+1,\beta+1) = \int_0^1 t^\alpha(1-t)^\beta dt, \quad \text{Beta function}$$

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t} dt, \quad \text{Gamma function}$$

$$\sqrt{\pi} = \int_{-\infty}^\infty e^{-t^2} dt, \quad \text{Gaussian integral}$$

Note: these functions are hypergeometric functions on $G_{2,3}$

Introduction

3-terms recurrence relations

$$tp_n(t) = p_{n+1}(t) + \alpha_n p_n(t) + \beta_n p_{n-1}(t).$$

It is important to know α_n, β_n.

- Put

  $$D_n = \det\left(\int_I t^{j+k} w(t)dt\right)_{j,k=0}^{n-1}$$

  Hankel determinant whose elements are moments of $w(t)$.
- Selberg type integral representation

  $$D_n = \frac{1}{n!}\int_I \cdots \int_I \prod_{i<j}(t_i - t_j)^2 \prod_{k=1}^{n} w(t_k)dt_1 \cdots dt_n,$$

- $\beta_n = \frac{D_{n-1}D_{n+1}}{D_n^2}$ and α_n is also computable in terms of $\{D_m\}$

Introduction

Relation to Painleve equation.

Take

- $w(t) = t^{\alpha}(1-t)^{\beta}(t-x)^{\gamma}$,
- $D_n(x) = \frac{1}{n!}\int_I \cdots \int_I \prod_{i<j}(t_i - t_j)^2 \prod_{k=1}^{n} t_k^{\alpha}(1-t_k)^{\beta}(t_k - x)^{\gamma} dt$

Theorem(Dai-Zhang)

$$H_n(x) = x(x-1)\frac{d}{dx}\log D_n(x) + d_1 x + d_2$$

$$d_1 = -n(n+\alpha+\beta+\gamma) - \frac{(\alpha+\beta)^2}{4},$$

$$d_2 = -\frac{1}{4}\left[2n(n+\alpha+\beta+\gamma) + \beta(\alpha+\beta) - \gamma(\alpha-\beta)\right],$$

then $H_n(x)$ satisfies Okamoto's σ-form equation for P6.

Other Painlevés and weight functions.

- P5: $t^\alpha(1-t)^\beta e^{-x/t}$, $(1+t)^\alpha(1-t)^\beta e^{-xt}$:Kummer
- P4: $|t-x|^\alpha e^{-t^2}$, $t^\alpha e^{-t^2+xt}$, $|t^{2\alpha+1}|e^{-t^4+xt^2}$, : Hermite-Weber
- P3: $t^\alpha e^{-t-x/t}$: Bessel
- P2: $e^{t^3/3+xt}$: Airy

They are all integrands of integral rep. of HGF on $G_{2,4}$

Question

If we replace the above $w(t)$ with the integrand of general hypergeometric function on $G_{2,N}$, what we can say?

Introduction

We want to discuss

- Hypergeometric functions on $G_{2,N}$
- Generalized anti-self dual Yang-Mills equation(GYM)
- GYM+symmetry of Jordan group$\to$Schlesinger system and confluent type system(GSS)
- Ward Ansatz solution $\to$ particular sol of 2×2 GSS with GHGF
- Relation to semi-classical orthogonal polynomial theory

Hypergeometric functions(1)

- $\lambda = (n_1, \dots, n_\ell)$:a partition of N.
- $Z_\lambda = \{z \in \mathrm{Mat}(2, N) \mid \mathrm{rk}\, z = 2 \ + \text{some condtion}\}$
- H_λ: maximal abelian subgroup of $\mathrm{GL}(N)$

$$H_\lambda = J(n_1) \times \cdots \times J(n_\ell) \tag{1}$$

$$J(n) = \left\{ \begin{pmatrix} h_0 & h_1 & \dots & h_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_1 \\ & & & h_0 \end{pmatrix} \in \mathrm{GL}(n). \right\}$$

- $\chi(\cdot, \alpha) : \tilde{H}_\lambda \to \mathbb{C}^\times$: a character. ($\chi_{(1^N)}(h, \alpha) = h_1^{\alpha_1} \cdots h_N^{\alpha_N}$)
- GHGF: $F(z) = \int_C \chi(\vec{t}z, \alpha) dt$, where

$$z = (z_1, \dots, z_N) \in Z, \quad \vec{t} = (1, t), \quad \vec{t}z = (\vec{t}z_1, \dots, \vec{t}z_N)$$

Hypergeometric functions(continued)

Description of $\chi(\cdot,\alpha)$:

Let $\theta_m(x)$ $(m \geq 0)$ be defined by

$$\sum_{0\leq m<\infty} \theta_m(x) T^m = \log(x_0 + x_1 T + x_2 T^2 + \cdots) \tag{2}$$

Then $\theta_0(x) = \log x_0$

$$\theta_1(x) = \frac{x_1}{x_0}$$

$$\theta_2(x) = \frac{x_2}{x_0} - \frac{1}{2}\left(\frac{x_1}{x_0}\right)^2$$

$$\theta_3(x) = \frac{x_3}{x_0} - \left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right) + \frac{1}{3}\left(\frac{x_1}{x_0}\right)^3$$

$$\vdots$$

Hypergeometric function (continued)

A character $\chi_\lambda : \tilde{H}_\lambda \to \mathbb{C}^\times$ is given by

$$\chi_\lambda(h;\alpha) = \prod_{1\le k\le \ell} \exp\left(\sum_{0\le i<n_k} \alpha_i^{(k)}\theta_i(h^{(k)})\right),$$

for

$$h = (h^{(1)},\cdots,h^{(\ell)}) \in \tilde{H}_\lambda,\ h^{(k)} \in \tilde{J}(n_k)$$

with some weight

$$\alpha = (\alpha^{(1)},\dots,\alpha^{(\ell)}) \in \mathbb{C}^N, \alpha^{(k)} = (\alpha_0^{(k)},\dots,a_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$$

.

Hypergeometric functions (2)

$F(z)$ satisfies

- $F(gz) = \det(g^{-1})F(z), \quad g \in \mathrm{GL}(2)$
- $F(zh) = \chi(h, \alpha)F(z), \quad h \in \tilde{H}_\lambda$
- $(\partial_{0j}\partial_{1k} - \partial_{0k}\partial_{1j})F = \begin{vmatrix} \partial_{0j} & \partial_{0k} \\ \partial_{1k} & \partial_{1k} \end{vmatrix} F = 0, \quad j \neq k,$

where $z = (z_{ij})_{0 \le i \le 1, 1 \le j \le N}$ and $\partial_{ij} = \partial/\partial z_{ij}$.

Hypergeometric functions, Example

- $\lambda=(1,1,1,1)$: Gauss HGF
  - $\chi(h)=h_1^{\alpha_1}\cdots h_4^{\alpha_4}$ with $\alpha_1+\alpha_2+\alpha_3+\alpha_4=-2$
  - $F(z)=\int_C(z_{01}+z_{11}t)^{\alpha_1}\cdots(z_{04}+z_{14}t)^{\alpha_4}dt$
  - $z\to\begin{pmatrix}1&0&1&-x\\0&1&-1&1\end{pmatrix}$, $F=\int_C t^{\alpha_2}(1-t)^{\alpha_3}(t-x)^{\alpha_4}dt$
- $\lambda=(2,1,1)$: Kummer's confluent HGF
  - $\chi(h)=h_1^{\alpha_1}e^{\alpha_2\frac{h_2}{h_1}}h_3^{\alpha_3}h_4^{\alpha_4}$ with $\alpha_1+\alpha_3+\alpha_4=-2$
  - $F(z)=\int_C(z_{01}+z_{11}t)^{\alpha_1}\exp\left(\alpha_2\frac{z_{02}+z_{12}t}{z_{01}+z_{11}t}\right)\prod_{i=3,4}(z_{0i}+z_{1i}t)^{\alpha_i}dt$
  - When $z\to\begin{pmatrix}1&0&0&1\\0&x&1&-1\end{pmatrix}$, $F=\int_C e^{-xt}t^{\alpha_3}(1-t)^{\alpha_4}dt$
- $\lambda=(2,2)\leftrightarrow$Bessel
- $\lambda=(3,1)\leftrightarrow$Hermite-Weber
- $\lambda=(4)\leftrightarrow$Airy

Generalized Yang Mills

- $U \subset Z = \{z \in \mathrm{Mat}(2, N) \mid rk\, z = 2\}$: open
- D a holo. connection on $U \times \mathbb{C}^r$:

$$D = d + \sum_{i,j} \Phi_{ij}(z) dz_{ij} = \sum_{ij} D_{ij} dz_{ij}$$

$$D_{ij} = \frac{\partial}{\partial z_{ij}} + \Phi_{ij}(z), \quad \Phi_{ij}(z) \in \mathfrak{gl}(r)$$

Definition

A holo connection D is GYM, if

$$[\zeta D_{0j} - D_{1j}, \zeta D_{0k} - D_{1k}] = 0 \quad \forall j \neq k, \forall \zeta \in \mathbb{C} \tag{3}$$

Equivalently, nonlinear equations for $\Phi_{ij}(z)$:

$$[D_{0j}, D_{0k}] = 0, \quad [D_{1j}, D_{1k}] = 0, \quad [D_{0j}, D_{1k}] + [D_{1j}, D_{0k}] = 0 \tag{4}$$

GYM:Geometric meaning

:

- $\mathbb{P}\mathcal{C} := \{([\zeta_0, \zeta_1], z) \in \mathbb{P}^1 \times Z \mid (\zeta_0, \zeta_1) \neq (0, 0)\}$:Corresp. sp.
- Double fibration:

$$\begin{array}{ccc} & \mathbb{P}\mathcal{C} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^{N-1} & & Z \end{array}$$

$$\pi_1([\zeta_0, \zeta_1], z) = [\zeta_0 \vec{z_0} + \zeta_1 \vec{z_1}], \quad z = \begin{pmatrix} \vec{z_0} \\ \vec{z_1} \end{pmatrix}$$

$$\pi_2([\zeta_0, \zeta_1], z) = z$$

- $z \in Z \to \hat{z} := \pi_1(\pi_2^{-1}(z)) = \{[\zeta_0 \vec{z_0} + \zeta_1 \vec{z_1}]\}$: line in $\mathbb{P}^{N-1}$joining 2 points $[\vec{z_0}], [\vec{z_1}]$ called twistor line
- $p = [x] \in \mathbb{P}^{N-1} \to \tilde{p} := \pi_2(\pi_1^{-1}([x])) = \{z \mid \vec{z_0} \wedge \vec{z_1} \wedge x = 0\}$: twistor surface dim $= N - 1$

GYM:Geometric meaning

Fact

A connection D on $U \times \mathbb{C}^r$ is GYM
$\Leftrightarrow D|_{\tilde{p}}$ is integrable for $\forall p \in \hat{U} = \pi_1(\pi_2^{-1}(U)) \subset \mathbb{P}^{N-1}$.

Ward-Penrose transform

Important aspect of twistor theory is Ward-Penrose transform

- $U \subset Z$: an open subset (with certain additional condition)
- $\hat{U} = \pi_1(\pi_2^{-1}(U))$.

Ward-Penrose transform:

$$\left\{ \begin{array}{l} \text{holo. vect bundle} \\ E \to \hat{U} \text{ of } rk = r, \\ \text{trivial on twistor} \\ \text{lines } \hat{q}(q \in U) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{solutions } D \text{ of} \\ \text{GYM on } U \end{array} \right\}$$

- Twistor bundle:=vect. bundle $E \to \hat{U}$ trivial on twistor lines

Ward-Penrose tranform(continue)

Transform from twistor bundle to solution of GYM

- $E \to \hat{U}$: $\mathrm{SL}(2)$ twistor bundle,
- $\pi_1^* E$: lift of E to $\pi_1^{-1}(\hat{U}) = \mathbb{P}^1 \times U$ with

$$F^* = F(\vec{z_0} + \zeta \vec{z_1}) \quad \text{lift of the patching matrix for } E$$

- E is trivial on twistor lines $\to$ Riemann-Hilbert decomposition $F^* = \tilde{f}^{-1} \cdot f$ with
  - $f \in \mathrm{SL}(2)$: holo on $V = \{|\zeta| < 2\} \times U$
  - $\tilde{f} \in \mathrm{SL}(2)$: holo on $\tilde{V} = \{|\zeta| > 1\} \times U$ with $\tilde{f}(\infty) = 1_2$
- $F^*$is a lift of F $\to$

$$(\partial_{1j} - \zeta \partial_{0j}) F^* = (\partial_{1j} - \zeta \partial_{0j})(\tilde{f}^{-1} \cdot f) = 0$$

$$\rightsquigarrow \partial_{1j} \tilde{f} \cdot \tilde{f}^{-1} - \zeta \partial_{0j} \tilde{f} \cdot \tilde{f}^{-1} = \partial_{1j} f \cdot f^{-1} - \zeta \partial_{0j} f \cdot f^{-1}$$

Ward-Penrose transform(continue)

- By Liouville Th., both sides define a linear fct. of ζ with pole at $\zeta = \infty$ of order at most 1 and $\tilde{f}(\infty) = 1_2 \rightsquigarrow$

$$\begin{aligned}\partial_{1j}\tilde{f}\cdot\tilde{f}^{-1} - \zeta\partial_{0j}\tilde{f}\cdot\tilde{f}^{-1} = \partial_{1j}f\cdot f^{-1} - \zeta\partial_{0j}f\cdot f^{-1} = -\Phi_{1j}(z)\\ \Leftrightarrow [(\partial_{1j} + \Phi_{1j}(z)) - \zeta(\partial_{0j} + 0)]f = 0,\\ [(\partial_{1j} + \Phi_{1j}(z)) - \zeta(\partial_{0j} + 0)]\tilde{f} = 0\end{aligned}$$

- $\nabla = d + \sum_j \Phi_{1j}(z)dz_{1j}$ is a sol of GYM
- Note that, if $f = f_0(z) + f_1(z)\zeta + \cdots$, then Φ_{1j} can be determined only from f_0;

$$\Phi_{1j}(z) = -\partial_{1j}f_0 \cdot f_0^{-1}$$

Isomonodromic deformation

<u>Notation</u>

$x = (x_1, \ldots, x_N)$ homog. coordinate of $\mathbb{P}^{N-1}$; also use
$x = (x^{(1)}, \ldots, x^{(\ell)})$, $x^{(k)} \in \mathbb{C}^{n_k}$
For $\xi \in \mathfrak{h}_\lambda$, vector fields X_ξ on $\mathbb{P}^{N-1}$, Y_ξ on $\mathbb{P}\mathcal{C}$ is defined by

$$X_\xi g := \frac{d}{ds} g(x \exp s\xi)|_{s=0}, \quad Y_\xi h := \frac{d}{ds} h(([\zeta_0, \zeta_1], z \exp s\xi))|_{s=0}$$

<u>Def:</u> $\mathrm{SL}(2)$-twistor bdl E is symmetric w.r.t. H_λ
$\Leftrightarrow$ Infinitesimal action of $H_\lambda$on $\mathbb{P}^{N-1}$ can be lifted to E
$\Leftrightarrow$ $\exists$Lie derivation $\mathcal{L}_\xi (= X_\xi + A_\xi(x)$ in local form) acting on local sections of E, s.t. $\xi \mapsto \mathcal{L}_{X_\xi}$ is a Lie alg. hom.

$$\omega = \sum_{k=1}^{\ell} \sum_{j=0}^{n_k - 1} A_j^{(k)}(x) d\theta_j(x^{(k)})$$

Isomonodromic deformation

If $\mathrm{SL}(2)$-bdl $E \to \hat{U}$ is symmetric w.r.t. H_λ
$\to$Lie deriv. $\mathcal{L}_\xi$ induces an integrable con. $\nabla(= d + \omega\ )$ on $\hat{U}$
$\to$ the lift ∇^* of ∇ to $\pi_1^* E \to \pi_1^{-1}(\hat{U}) \subset \mathbb{P}^{N-1}$ must satisfy

$$Y_\xi F^* = F^* A_\xi - \tilde{A}_\xi F^* \qquad (\forall \xi \in \mathfrak{h}_\lambda)$$

where$A_\xi$and $\tilde{A}_\xi$ comes from the Lie der. induced from $\mathcal{L}_\xi$on V and $\tilde{V}$
$\to$Using R-H $F^* = \tilde{f}^{-1} f$ with $\tilde{f}(\infty) = 1_2$, trivialize the bdl on the fiber $\mathbb{P}^1$,

$$\nabla^* = d + f\omega f^{-1} - df \cdot f^{-1} = d + \tilde{f}\tilde{\omega}\tilde{f}^{-1} - d\tilde{f} \cdot \tilde{f}^{-1}$$

This $\nabla *$ gives an isomono deform. and $(\nabla^*)^2 = 0$ gives a general Schlesinger system(GSS).
Remark: conn. form of ∇^*can be computed only from $A_\xi, f(0)$.
When $\lambda = (1^N)$, above construction gives Schlesinger equation.

Ward Ansatz solution of GYM

Objective

Construct particular sol. of GSS by HGF

- $U \subset Z$: open set.
- $\hat{U} = \pi_1(\pi_2^{-1}(U)) \simeq \mathbb{P}^1 \times U$.
- Ansatz:
  - E: $\mathrm{SL}(2)$-twistor bundle on $\hat{U}$ corresponding to a sol. of GYM.
  - the transition function F^* of $\pi_1^* E$ has the form

$$F^* = \begin{pmatrix} \zeta^m & \phi(\zeta, z) \\ & \zeta^{-m} \end{pmatrix} \text{ on } V \cap \tilde{V},$$

  - $\phi(\zeta, z) = \sum_{n=-\infty}^{\infty} \phi_n(z) \zeta^{-n}$: Laurent expansion

Ward Ansatz solution of GYM

- E is trivial on twistor lines $\Rightarrow$

$$F^* = \tilde{f}^{-1} \cdot f \tag{5}$$

with f holo. in V and $\tilde{f}$ holo. in $\tilde{V}$ with $\tilde{f}(\infty) = 1_2$. We can determine f and $\tilde{f}$ by (5) using linear algebra:

$$f(0) = \frac{(-1)^m}{\tau_m^0} \begin{pmatrix} \tau_m^1 & \tau_{m+1}^0 \\ \tau_{m-1}^0 & \tau_m^{-1} \end{pmatrix} \in \mathrm{SL}(2)$$

$$\tau_m^0 = \begin{vmatrix} \phi_{1-m} & \phi_{2-m} & \cdots & & \phi_0 \\ \phi_{2-m} & & & \phi_0 & \\ \vdots & & & & \vdots \\ & \phi_0 & & & \phi_{m-2} \\ \phi_0 & & \cdots & \phi_{m-2} & \phi_{m-1} \end{vmatrix}$$

Ward Ansatz solution of GYM

How we determine $\phi_n(z)$ in F^* ?

F^* is a lift of a transition fct. F of E
$\Rightarrow (\partial_{1j} - \zeta\partial_{0j})F^* = 0,$
$\Rightarrow (\partial_{1j} - \zeta\partial_{0j})\phi = 0$

$$\begin{aligned}
&\Leftrightarrow (\partial_{1j} - \zeta\partial_{0j}) \sum_n \phi_n(z)\zeta^{-n} = 0 \\
&\Leftrightarrow \ \partial_{1j}\phi_{n-1} = \partial_{0j}\phi_n \qquad (6) \\
&\Rightarrow \ (\partial_{0j}\partial_{1k} - \partial_{0k}\partial_{1j})\phi_n = 0 \quad (j \neq k):
\end{aligned}$$

(the equations for the image of Radon transform).

Ward Ansatz for GSS

Determine ϕ_n so that the Ward Ansatz solution becomes a particular solution of GSS.

- $\lambda = (n_1, \dots, n_\ell)$: a partition of N, and $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$: a character
- Take $n_0 \in \mathbb{Z}$ and put

$$\phi_n(z) = \int_C t^{n-n_0} \chi(\vec{t}z, \alpha) dt.$$

- $\phi_n(z)$satisfies contiguous relation $\partial_{1j}\phi_{n-1} = \partial_{0j}\phi_n$ and W

$$\phi_n(zh) = \phi(z)\chi(h, \alpha)$$

- Corresponding F^* gives a connection corresponding to a solution of GSS, if there exist $\tilde{A}_\xi$: holomorphic in $\tilde{V}$ and A_ξ : holomorphic in V s.t.

$$X_\xi F^* = F^* A_\xi - \tilde{A}_\xi F^* \quad (\xi \in \mathfrak{h}_\lambda).$$

Ward Ansatz for GSS(continue)

- We can take

$$A_\xi = \frac{1}{2}\begin{pmatrix} \langle \xi, \alpha \rangle & \\ & -\langle \xi, \alpha \rangle \end{pmatrix}, \quad \tilde{A}_\xi = -A_\xi$$

- Thus gives a particular sol of GSS in terms of GHGF