On the center manifold of unfolded complex saddle-node singularities

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Unfolded saddle-node singularity

A parametric family of singular ODE's

$$(x^2 - \epsilon)\frac{dy}{dx} = M\,y + F(x,\epsilon,y), \quad (x,\epsilon,y) \in (\mathbb{C}\times\mathbb{C}\times\mathbb{C}^m, 0) \tag{1}$$

with M *invertible* and $F'_y(0,0,0) = 0$.

Bounded solutions $y(x,\sqrt{\epsilon})$ *of* (1) *near the singular points* $x = \pm\sqrt{\epsilon}$

$\leftrightarrow$ *Center manifold of an unfolded saddle-node singularity*

$$\dot{x} = x^2 - \epsilon, \qquad \dot{y} = M\,y + F(x,\epsilon,y). \tag{2}$$

In general, an analytic center-manifold does not exist.

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Theorem (Formal solution)

The equation (1) *possesses a unique formal solution* $\hat{y}(x, \epsilon) = \sum_{k,l=1}^{+\infty} y_{kl} x^k \epsilon^l$, $\|y_{kl}\| \leq CK^{k+2l} \cdot (k+2l)!,\ C, K > 0.$

For $\epsilon = 0$:

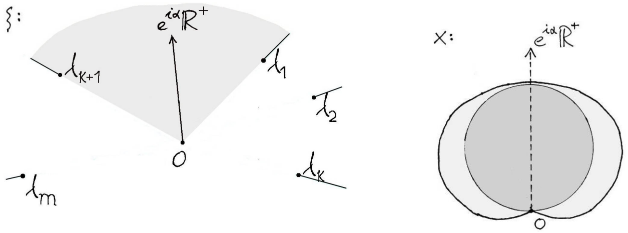
Theorem (Braaksma, Écalle, Hukuhara, Malmquist, Ramis, Sibuya, ...)

The formal series

$$\hat{y}_0(x) = \sum_{k=1}^{+\infty} y_{k0} x^k$$

is Borel summable in each direction α *with* $e^{i\alpha}\mathbb{R}^+$ *disjoint from* $\operatorname{Spec}(M)$.

$$\hat{y}_0(x) \;\mapsto\; \hat{B}[\hat{y}_0](\xi) := \sum_{k=1}^{+\infty} \frac{y_{k0}}{(k-1)!}\xi^{k-1} \;\mapsto\; y_0^{\alpha}(x) := \int_0^{+\infty e^{i\alpha}} \hat{\mathcal{B}}[\hat{y}_0](\xi) e^{-\frac{\xi}{x}}\, d\xi.$$



Confluence of local solutions

Linearization of the vector field (2) at $x = \pm\sqrt{\epsilon}$

$$\dot{x} = \pm 2\sqrt{\epsilon}(x - \sqrt{\epsilon}), \qquad \dot{y} = M_{\pm\sqrt{\epsilon}}.$$

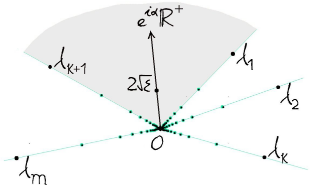
For $\epsilon \neq 0$, if

$$\pm 2\sqrt{\epsilon}\mathbb{N}_{>0} \cap \mathrm{Spec}\,(M_{\pm\sqrt{\epsilon}}) = \emptyset,$$

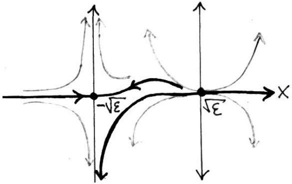
then there is a local analytic solution $y_{\pm\sqrt{\epsilon}}(x)$ at $x = \pm\sqrt{\epsilon}$.

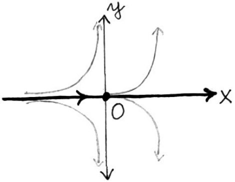
Theorem (Glutsyuk)

When $\epsilon \to 0$ radially with $\pm 2\sqrt{\epsilon} \in e^{i\alpha}\mathbb{R}^+$, then $y_{\pm\sqrt{\epsilon}} \to y_0^\alpha$.



The general mismatch of the local solutions $y_{+\sqrt{\epsilon}}(x)$, $y_{-\sqrt{\epsilon}}(x)$ persisting to the limit explains the divergence of the limit formal solution $\hat{y}_0(x)$.



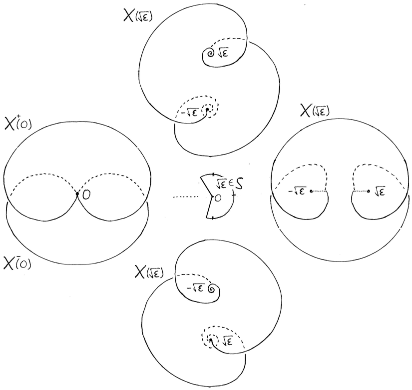


Theorem

Let a straight line $e^{i\alpha}\mathbb{R}$ divide $\mathrm{Spec}\,(M)$ in two parts.
There is a unique bounded and analytic solution $y^{\alpha}(x,\sqrt{\epsilon})$ defined on a ramified domain

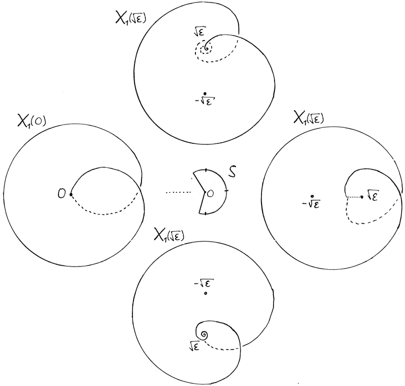
$$X = \coprod_{\sqrt{\epsilon}\in S} X(\sqrt{\epsilon})$$

in the $(x,\sqrt{\epsilon})$-space, such that $y^{\alpha}(x,0) \upharpoonright X^{\pm}(0)$ are the respective Borel sums of $\hat{y}_0$ in the directions α, resp. $\alpha+\pi$.



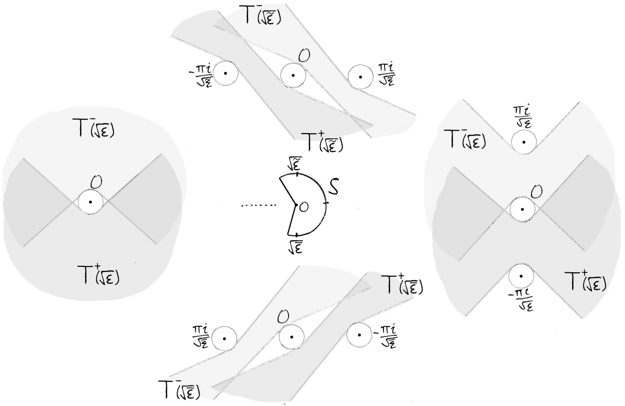
If, moreover, the spectrum of the matrix M is of *Poincaré type*, i.e. the whole $\mathrm{Spec}\, M$ lies on the same side of a line $e^{i\alpha}\mathbb{R}$, then the corresponding solution $y^{\alpha}(x, \sqrt{\epsilon})$ is ramified only at one of the singular points, and analytic at the other.

Such is the case in dimension $m = 1$.



The domain in the time coordinate of the vector field $-(x^2-\epsilon)\partial_x$

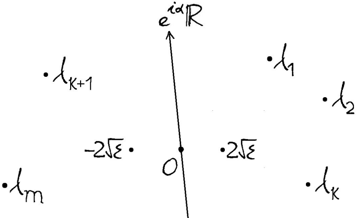
$$t(x,\epsilon) = -\int \frac{dx}{x^2-\epsilon} := \begin{cases} -\frac{1}{2\sqrt{\epsilon}} \log \frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}}, & \text{if} \quad \epsilon \neq 0, \\ \frac{1}{x}, & \text{if} \quad \epsilon = 0, \end{cases}$$



Construction I: Hadamard–Perron

Theorem (Hadamard-Perron)

Suppose that the spectrum of linearization matrix of a holomorphic vector field at a singularity consists of two parts separated by a line through origin. Then the vector field has two holomorphic invariant manifolds tangent to the corresponding eigenspaces of the linearization.



$(m-k+1)$-dimensional invariant manifold at $(x, y) = (-\sqrt{\epsilon}, 0)$

$(k+1)$-dimensional invariant manifold at $(x, y) = (\sqrt{\epsilon}, 0)$

The two manifolds intersect transversally along the graph of $y^{\alpha}(x, \sqrt{\epsilon})$.

($\sqrt{\epsilon}$ can vary in a sector S of opening $> \pi$).

Construction II: Unfolded Borel-Laplace transformations

$$y^{\alpha}(x,\sqrt{\epsilon}) = \int_{e^{i\alpha}\mathbb{R}} u^{\alpha}(\xi,\sqrt{\epsilon})\, e^{-t(x,\epsilon)\xi}\, d\xi \quad \to \quad y^{\alpha}(x,0) = \int_{0}^{+\infty e^{i\alpha}} u^{\alpha}(\xi,0)\, e^{-\frac{\xi}{x}}\, d\xi,$$

where $u^{\alpha}(\xi,\sqrt{\epsilon})$ is a solution to some convolution equation. The integral converges on a strip between the points 0 and $\pm\frac{\pi i}{\sqrt{\epsilon}}$ in the $t(x,\epsilon)$-coordinate in direction $-\alpha + \frac{\pi}{2}$.

The bilateral integral converges to a unilateral due to a factor in u^{α}

$$\chi_{\alpha}^{+}(\xi,\sqrt{\epsilon}) := \frac{1}{1 - e^{\frac{\xi\pi i}{\sqrt{\epsilon}}}} \quad \to \quad \chi_{\alpha}^{+}(\xi,0) := \begin{cases} 1, & \text{if } \ \xi \in]0,+\infty e^{i\alpha}[, \\ 0, & \text{if } \ \xi \in]-\infty e^{i\alpha},0[. \end{cases}$$

Construction III: Borel sum of $\hat{y}(x, \sqrt{\epsilon})$

Blow-up $x = \sqrt{\epsilon} z$ $\rightsquigarrow$ Borel summation of $\hat{y}(\sqrt{\epsilon} z, \epsilon)$ in $\sqrt{\epsilon}$ $\rightsquigarrow$ blow-down:

$$y^{\alpha}(x, \sqrt{\epsilon}) = \int_0^{+\infty e^{i\theta}} U(sx, s^2\epsilon)\, e^{-s}\, ds, \qquad \theta \in]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[$$

where $U(x, \epsilon)$ is analytic extension of the sum

$$U(x, \epsilon) = \sum_{j,k} \frac{y_{kj}}{(k+2j)!} x^k \epsilon^j.$$

Corollary

y^{α} and $\hat{y}$ satisfy the same $\partial_x, \partial_\epsilon$-differential relations over $\mathbb{C}\{x, \sqrt{\epsilon}\}$.

Application: Linear systems (Lambert, Rousseau)

$$(x^2 - \epsilon)\frac{dy}{dx} - A(x,\epsilon)y = 0,$$

Suppose the eigenvalues of $A(0,0)$ are pairwise disjoint.

Is there an invertible linear transformation $y = T\,\psi$, such that ϕ satisfies

$$(x^2 - \epsilon)\frac{d\psi}{dx} - \Lambda(x,\epsilon)\psi = 0,$$

with $\Lambda(x,\epsilon) = \text{Diag}(\lambda_j(x,\epsilon))$ diagonal?

$\lambda_j(x,\epsilon) = \lambda_j^{(0)}(\epsilon) + x\lambda_j^{(1)}(\epsilon)$ are the eigenvalues of $A(x,\epsilon)$ modulo $(x^2 - \epsilon)$.

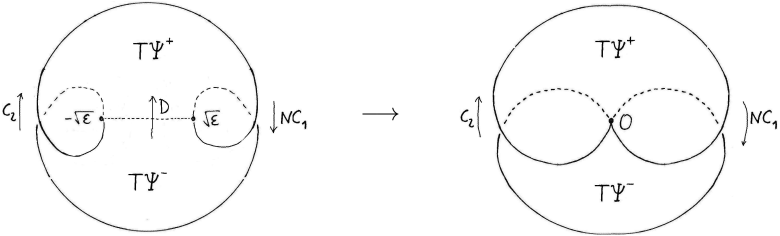
The matrix T satisfies

$$(x^2 - \epsilon)\frac{dT}{dx} = AT - T\Lambda,$$

whose projectivization reduces to (1).

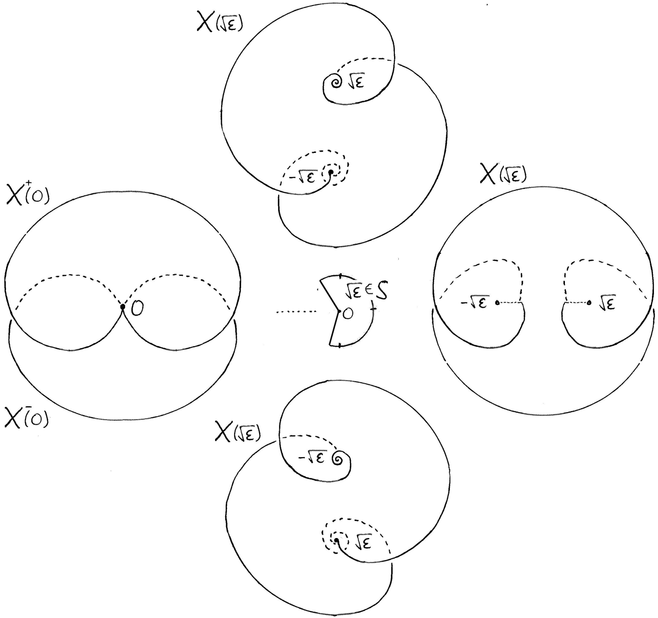
Fundamental matricial solution $Y(x,\sqrt{\epsilon}) = T(x,\sqrt{\epsilon})\Psi(x,\epsilon)$ with $\Psi(x,\epsilon) = \mathrm{Diag}(\phi_j(x,\epsilon))$

$$\psi_j(x,\epsilon) = e^{\int \frac{\lambda_j(x,\epsilon)}{x^2-\epsilon}\,dx} = \left(\frac{x-\sqrt{\epsilon}}{x+\sqrt{\epsilon}}\right)^{\frac{\lambda_j^{(0)}(\epsilon)}{2\sqrt{\epsilon}}} (x^2-\epsilon)^{\frac{\lambda_j^{(1)}(\epsilon)}{2}} \quad \rightarrow \quad e^{-\frac{\lambda_j^{(0)}(0)}{x}} x^{\lambda^{(1)}(0)}.$$



$N = \mathrm{Diag}\left(e^{2\pi i \lambda_j^{(1)}}\right)$, $D = \mathrm{Diag}\left(e^{\pi i (\frac{\lambda_j^{(0)}}{\sqrt{\epsilon}} - \lambda_j^{(1)})}\right)$ ... *formal monodromy matrices*, C_1, C_2 ... *"Stokes matrices"*.

There is a natural flag structure on the solution space on each of the two intersection sectors, invariant w.r.t. bounded automorphisms, given by their asymptotic order of growth (corresponding to different growths of $|\psi_j(x,\epsilon)|$, $j = 1,\ldots,n$).


$X(\sqrt{\varepsilon})$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
$X^+(0)$
0
$X^-(0)$
$\sqrt{\varepsilon} \in S$
0
$X(\sqrt{\varepsilon})$
$-\sqrt{\varepsilon}$
$\sqrt{\varepsilon}$
$X(\sqrt{\varepsilon})$
$-\sqrt{\varepsilon}$
$\sqrt{\varepsilon}$

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