On Sobolev and potential spaces related to Jacobi expansions

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Plan of the talk

0. Jacobi setting
1. **Potential spaces**
2. **Higher order derivatives**
3. **Sobolev spaces**
4. **First main result**
5. **Methodology of the proof**
6. **Fractional g-functions**
7. **Second main result**
8. **Applications**

Jacobi setting

The classical Jacobi polynomials are defined by

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big]$$

It is natural and convenient to consider also the normalized trigonometric Jacobi polynomials

$$\mathcal{P}_n^{\alpha,\beta}(\theta) = c_n^{\alpha,\beta} P_n^{\alpha,\beta}(\cos\theta)$$

The Jacobi 'functions' are defined by

$$\phi_n^{\alpha,\beta}(\theta) := \Big(\sin\frac{\theta}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\Big)^{\beta+1/2}\mathcal{P}_n^{\alpha,\beta}(\theta), \qquad \theta\in(0,\pi)$$

- $\{\phi_n^{\alpha,\beta} : n \geq 0\}$ is an ONB in $L^2(0,\pi)$

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- $\{\phi_n^{\alpha,\beta} : n \geq 0\}$ is an ONB in $L^2(0,\pi)$

Jacobi operator

- $\phi_n^{\alpha,\beta}$ are eigenfunctions of the Jacobi operator

$$\boldsymbol{L}_{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{1-4\alpha^2}{16\sin^2\frac{\theta}{2}} - \frac{1-4\beta^2}{16\cos^2\frac{\theta}{2}}$$

  with eigenvalues $\lambda_n^{\alpha,\beta} = \left(n + \frac{\alpha+\beta+1}{2}\right)^2$

- we have the factorization

$$\boldsymbol{L}_{\alpha,\beta} = D^*_{\alpha,\beta} D_{\alpha,\beta} + \lambda_0^{\alpha,\beta}$$

  where

$$D_{\alpha,\beta} = \frac{d}{d\theta} - \frac{2\alpha+1}{4}\cot\frac{\theta}{2} + \frac{2\beta+1}{4}\tan\frac{\theta}{2}$$

  and $D^*_{\alpha,\beta}$ is the formal adjoint of $D_{\alpha,\beta}$ in $L^2(0,\pi)$

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Jacobi-Poisson semigroup

The Jacobi-Poisson semigroup $H_t^{\alpha,\beta} = \exp(-tL_{\alpha,\beta}^{1/2})$ has the integral representation

$$H_t^{\alpha,\beta} f(\theta) = \int_0^\pi H_t^{\alpha,\beta}(\theta,\varphi) f(\varphi)\, d\varphi, \qquad t > 0, \quad \theta \in (0,\pi)$$

with the Jacobi-Poisson kernel

$$H_t^{\alpha,\beta}(\theta,\varphi) = \sum_{n=0}^{\infty} \exp\Big(- t\sqrt{\lambda_n^{\alpha,\beta}}\Big)\, \phi_n^{\alpha,\beta}(\theta)\, \phi_n^{\alpha,\beta}(\varphi)$$

Pencil phenomenon

- if $\alpha, \beta \geq -1/2$, then $\phi_n^{\alpha,\beta} \in L^p(0,\pi)$, $1 \leq p \leq \infty$
- if $\alpha < -1/2$ or $\beta < -1/2$, then $\phi_n^{\alpha,\beta} \in L^p(0,\pi)$ if and only if $p < p(\alpha,\beta) := -1/\min(\alpha + 1/2, \beta + 1/2)$

This leads to the restriction of L^p mapping properties of operators associated with $L_{\alpha,\beta}$ to $p \in E(\alpha,\beta)$, where

$$E(\alpha,\beta) := \begin{cases} (1,\infty), & \alpha,\beta \geq -1/2 \\ (p'(\alpha,\beta), p(\alpha,\beta)), & \text{otherwise} \end{cases}$$

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Potential operators

- $\sigma > 0$
- for $\alpha + \beta \neq -1$ we consider the Jacobi-Riesz potential operator $L_{\alpha,\beta}^{-\sigma}$
- if $\alpha + \beta = -1$, zero is the eigenvalue of $L_{\alpha,\beta}$!
- we consider instead the Jacobi-Bessel potential operator $(\mathrm{Id} + L_{\alpha,\beta})^{-\sigma}$
- both well defined spectrally and bounded on $L^2(0,\pi)$, possess integral representations valid far beyond $L^2(0,\pi)$

Proposition (*Nowak, Roncal, 2012*)

Let $\alpha, \beta > -1$ and let $\sigma > 0$. Assume that $p \in E(\alpha, \beta)$. Then

(a) $L_{\alpha,\beta}^{-\sigma}$ is bounded and one to one on $L^p(0,\pi)$ when $\alpha + \beta \neq -1$;

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Potential spaces

Given $s > 0$ and $p \in E(\alpha, \beta)$, we define

$$\mathcal{L}^{p,s}_{\alpha,\beta} := \begin{cases} L_{\alpha,\beta}^{-s/2}\big(L^p(0,\pi)\big), & \alpha+\beta \neq -1 \\ (\mathrm{Id} + L_{\alpha,\beta})^{-s/2}\big(L^p(0,\pi)\big), & \alpha+\beta = -1 \end{cases}$$

Then the formula

$$\|f\|_{\mathcal{L}^{p,s}_{\alpha,\beta}} := \begin{cases} \|L_{\alpha,\beta}^{s/2} f\|_{L^p}, & \alpha+\beta \neq -1 \\ \|(\mathrm{Id} + L_{\alpha,\beta})^{s/2} f\|_{L^p}, & \alpha+\beta = -1 \end{cases}$$

defines a norm on $\mathcal{L}^{p,s}_{\alpha,\beta}$.

- $\mathcal{L}^{p,s}_{\alpha,\beta}$ equipped with this norm is a Banach space

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Sobolev spaces

According to a general concept, for $m \geq 1$, Sobolev space should be defined by

$$\mathbb{W}^{p,m}_{\alpha,\beta} := \{ f \in L^p(0,\pi) : \mathbb{D}^{(k)} f \in L^p(0,\pi), k = 1, \dots, m \}$$

and equipped with the norm

$$\|f\|_{\mathbb{W}^{p,m}_{\alpha,\beta}} := \sum_{k=0}^{m} \|\mathbb{D}^{(k)} f\|_{L^p(0,\pi)}$$

- $\mathbb{D}^{(k)}$ are suitably defined differential operators of order k 'higher order derivatives'
- the differentiation is understood in a weak sense
- $\mathbb{W}^{p,m}_{\alpha,\beta}$ depends on a choice of $\mathbb{D}^{(k)}$!
- classical result: the isomorphism between Sobolev and potential spaces
- heart of the matter: how to define $\mathbb{D}^{(k)}$ so that the isomorphism is preserved in the Jacobi setting?

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- seemingly the most natural choice $\mathbb{D}^{(k)} = D^k_{\alpha,\beta}$ is not appropriate, since then the spaces $\mathbb{W}^{p,m}_{\alpha,\beta}$ and $\mathcal{L}^{p,m}_{\alpha,\beta}$ are not isomorphic in general
- another known notion of higher order 'derivative', supported by a good L^p-theory, is

$$\mathcal{D}^{(k)} := \underbrace{\ldots D_{\alpha,\beta} D^*_{\alpha,\beta} D_{\alpha,\beta} D^*_{\alpha,\beta} D_{\alpha,\beta}}_{k \text{ components}}$$

- ...but it does not work either!

Thus, inspired by Betancor et al. (2010), we introduce yet another higher order 'derivative'

$$D^{(k)} := D_{\alpha+k-1,\beta+k-1} \circ \ldots \circ D_{\alpha+1,\beta+1} \circ D_{\alpha,\beta}$$

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Main result

Define

$$W^{p,m}_{\alpha,\beta} := \left\{ f \in L^p(0,\pi) : D^{(k)} f \in L^p(0,\pi), k = 1, \ldots, m \right\}$$

Theorem (*B.L, 2013*)

Let $\alpha, \beta > -1$, $p \in E(\alpha, \beta)$ and $m \geq 1$. Then

$$W^{p,m}_{\alpha,\beta} = \mathcal{L}^{p,m}_{\alpha,\beta}$$

in the sense of isomorphism of Banach spaces.

Let

$$\mathcal{W}^{p,m}_{\alpha,\beta} := \{ f \in L^p(0,\pi) : \mathcal{D}^{(k)} f \in L^p(0,\pi), k = 1, \dots, m \}$$

Theorem (*B.L, 2013*)

Let $\alpha, \beta > -1$, $p \in E(\alpha, \beta)$ and $m \geq 1$. Then

$$\mathcal{L}^{p,m}_{\alpha,\beta} \subset \mathcal{W}^{p,m}_{\alpha,\beta}$$

in the sense of embedding of Banach spaces. However, the reverse inclusion does not hold for all parameters values. In particular, for each α, β satisfying $0 \neq \alpha, \beta < 1/p - 1/2$ there is $f \in \mathcal{W}^{p,2}_{\alpha,\beta}$ such that $f \notin \mathcal{L}^{p,2}_{\alpha,\beta}$.

- the set $S_{\alpha,\beta} := \text{span}\{\phi_n^{\alpha,\beta} : n \geq 0\}$ is a dense subspace of $W^{p,m}_{\alpha,\beta}$ and $\mathcal{L}^{p,m}_{\alpha,\beta}$; the proof uses the following:

Proposition (*B.L, 2013*)

Let $\alpha, \beta > -1$ and let $p \in E(\alpha, \beta)$. Then the maximal operator

$$H_*^{\alpha,\beta} f := \sup_{t>0} \left| H_t^{\alpha,\beta} f \right|$$

is bounded on $L^p(0, \pi)$.

- Riesz-Jacobi transforms $R^{k,1}_{\alpha,\beta} = D^{(k)} L^{-k/2}_{\alpha,\beta}$ and 'conjugate' Riesz-Jacobi transforms $R^{k,2}_{\alpha,\beta} = (D^{(k)})^* L^{-k/2}_{\alpha,\beta}$ are L^p-bounded
- there exist 'inverting' operators $T^k_{\alpha,\beta}$; one can write $f = T^k_{\alpha,\beta} R^{k,1}_{\alpha,\beta} R^{k,2}_{\alpha,\beta} f$ for suitable functions f

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- the set $S_{\alpha,\beta} := \operatorname{span}\{\phi_n^{\alpha,\beta} : n \geq 0\}$ is a dense subspace of $W_{\alpha,\beta}^{p,m}$ and $\mathcal{L}_{\alpha,\beta}^{p,m}$; the proof uses the following:

Proposition (*B.L, 2013*)

Let $\alpha, \beta > -1$ and let $p \in E(\alpha, \beta)$. Then the maximal operator

$$H_*^{\alpha,\beta} f := \sup_{t>0} \left| H_t^{\alpha,\beta} f \right|$$

is bounded on $L^p(0, \pi)$.

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Fractional square functions

The fractional square functions are given by

$$\mathfrak{g}^{\gamma}_{\alpha,\beta}(f)(\theta) = \left(\int_0^\infty \left| t^\gamma \partial_t^\gamma H_t^{\alpha,\beta} f(\theta) \right|^2 \frac{dt}{t} \right)^{1/2}$$

with the fractional derivative

$$\partial_t^\gamma F(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^\infty \frac{\partial^m}{\partial t^m} F(t+s)\, s^{m-\gamma-1}\, ds, \qquad t \in (0,\infty),$$

where $m = \lfloor \gamma \rfloor + 1$

g-functions

- auxiliary g-functions

$$\mathfrak{g}^{\gamma,k}_{\alpha,\beta}(f)(\theta) = \left(\int_0^\infty \left| t^{k-\gamma} \frac{\partial^k}{\partial t^k} H^{\alpha,\beta}_t f(\theta) \right|^2 \frac{dt}{t} \right)^{1/2}, \quad 0 < \gamma < k$$

- key identity

$$\mathfrak{g}^{\gamma,k}_{\alpha,\beta}(f) = \mathfrak{g}^{k-\gamma}_{\alpha,\beta}(L^{\gamma/2}_{\alpha,\beta} f), \qquad f \in \mathcal{L}^{\gamma,p}_{\alpha,\beta}$$

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Theorem (*B.L, 2014*)

Let $\alpha, \beta > -1$, $p \in E(\alpha,\beta)$, $\gamma > 0$ and assume that $\alpha+\beta \neq -1$. Then

$$\|f\|_p \simeq \|\mathfrak{g}^{\gamma}_{\alpha,\beta}(f)\|_p, \qquad f \in L^p(0,\pi)$$

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Furthermore,

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Second main result

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1. **Convergence of solutions to differential equations**

Given some initial data f we consider the following Cauchy problem:

$$\begin{cases} i\frac{\partial u(\theta,t)}{\partial t} = L_{\alpha,\beta}\, u(\theta,t), & \theta \in (0,\pi),\ t \in \mathbb{R} \\ u(\theta,0) = f(\theta) \end{cases}$$

- easy to check that $\exp(itL_{\alpha,\beta})f$ is a solution
- natural question: what regularity conditions to impose on f so that $\exp(itL_{\alpha,\beta})f$ tends to f a.e. as t tends to zero?

Theorem (*B.L, 2014*)

Let $\alpha,\beta > -1$ and $f \in W^{2,m}_{\alpha,\beta}$, $m > 0$. Then

$$\lim_{t\to 0} \exp(itL_{\alpha,\beta})f(\theta) = f(\theta)$$

for almost all $\theta \in (0,\pi)$.

Applications

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2. **Embedding theorems**

Theorem (*B.L, 2014*)

Let $\alpha, \beta > -1$, $m > 0$, $1 \leq p, q \leq \infty$ be such that the operator $L_{\alpha,\beta}^{-m/2}$ is strong type (p, q). Then $W_{\alpha,\beta}^{p,m} \subset L^q$ and

$$\|f\|_q \lesssim \|f\|_{W_{\alpha,\beta}^{p,m}}, \qquad f \in W_{\alpha,\beta}^{p,m}.$$

Furthermore, if p is such that $L_{\alpha,\beta}^{-m/2}$ is strong type (p, ∞), then $W_{\alpha,\beta}^{p,m} \subset C(0, \pi)$.

Remark: $L_{\alpha,\beta}^{-\sigma/2}$ is strong type (p, q), $1 \leq p, q \leq \infty$, iff

- in case $\alpha, \beta \geq -\frac{1}{2}$
  $\frac{1}{q} \geq \frac{1}{p} - \sigma \quad \text{and} \quad \left(\frac{1}{p}, \frac{1}{q}\right) \notin \{(1, 1-\sigma), (\sigma, 0)\}$
- in case $\alpha \wedge \beta < -\frac{1}{2}$
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Have a nice banquet!