Multi-level Gevrey solutions of singularly perturbed linear PDEs

Alberto Lastra[*], Stéphane Malek[○]

[*] Universidad de Alcalá, Spain
[○] Université Lille 1, France

Bedlewo, 18th September, 2015

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1. Study of ODEs and PDEs with complex perturbation parameter:

Motivation and related studies

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M. Canalis-Durand, J. Mozo-Fernández, R. Schäfke, Monomial summability and doubly singular differential equations, J. Differential Equations 233 (2007), no. 2, 485–511.

$$\epsilon^{\sigma} x^{r+1} \frac{d\boldsymbol{y}}{dx} = f(x, \epsilon, \boldsymbol{y}), \quad f(0, 0, \boldsymbol{0}) = \boldsymbol{0}.$$

The unique bivariate formal solution is monomially summable, in $\epsilon^{\sigma} x^{r}$.

Motivation and related studies

1. Study of ODEs and PDEs with complex perturbation parameter:

S. Kamimoto, T. Koike, On the Borel summability of 0-parameter solutions of nonlinear ordinary differential equations. Recent development of micro-local analysis for the theory of asymptotic analysis, 191–212, RIMS Kôkyûroku Bessatsu, B40, Res. Inst. Math. Sci. (RIMS), Kyoto, 2013.

Borel summability of $\lambda(t,\nu)=\lambda_0(t)+\nu^{-1}\lambda_1(t)+\dots$ of the equation

$$\frac{d^2\lambda}{dt^2}=\nu^2\frac{P(t,\lambda)}{Q(t,\lambda)}+\frac{R_1(t,\lambda,\dot{\lambda})}{R_2(t,\lambda)},$$

with $P(t,\lambda),Q(t,\lambda),R_2(t,\lambda)\in\mathbb{C}[t,\lambda]$, $R_1(t,\lambda,\dot{\lambda})\in\mathbb{C}[t,\lambda,\dot{\lambda}]$.

Motivation and related studies

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A. Lastra, S. Malek, J. Sanz, On the Gevrey solutions of threefold singular nonlinear partial differential equations. J. Differential Equations 255 (2013), no. 10, 3205–3232.

$$((z\partial_z+1)^{r_1}\epsilon^{r_3}(t^2\partial_t+t)^{r_2}+1)\partial_z^S X(t,z,\epsilon)$$
$$=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon)$$
$$+P(t,z,\epsilon,X(t,z,\epsilon)$$

Study of PDEs with Fuchsian and irregular singularities, and presence of a singular perturbation parameter ϵ.

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$$((z\partial_z + 1)^{r_1}\epsilon^{r_3}(t^2\partial_t + t)^{r_2} + 1)\partial_z^S X(t,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon) + P(t,z,\epsilon,X(t,z,\epsilon))$$

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- Studies in which multilevel Gevrey asymptotics appear:

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K. Suzuki, Y. Takei, Exact WKB analysis and multisummability - A case study -, RIMS Kôkyûroku, no. 1861, 2013, pp. 146–155.

Multisummability of WKB solutions of Schrödinger equation having both fixed and movable turning points.

Y. Takei, On the multisummability of WKB solutions of certain singularly perturbed linear ordinary differential equations, Opuscula Math. 35 (2015), no. 5, 775–802.

Multisummability of WKB solutions of

$$\left(\nu^{-m}\frac{d^m}{dz^m}+\nu^{-(m-1)}\frac{d^{m-1}}{dz^{m-1}+...q_m(z,\nu^{-1})}\right)\psi(z,\nu)=0.$$

Main problem

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Study formal and analytic solutions, and asymptotics related to

$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2}+a_2)(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1)\partial_z^S X(t,z,\epsilon)$$
$$= \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon),$$

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$r_1, r_2 \in \mathbb{N} := \{0,1,2,...\}$ $\quad$ $s_1, s_2 \in \mathbb{N}^\star := \mathbb{N} \setminus \{0\}$ $\quad$ $k \in \mathbb{N}, k \geq 2$

$a_1, a_2 \in \mathbb{C}^\star$

$\mathcal{S}$ finite subset of $\mathbb{N}^3$ $\quad$ $\kappa_1 < S$ for all $(s,\kappa_0,\kappa_1) \in \mathcal{S}$

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Main problem

Initial conditions

$$(\partial_z^j X)(t,0,\epsilon) = \phi_j(t,\epsilon) \in \mathcal{O}(\mathcal{T} \times \mathcal{E}), \qquad 0 \leq j \leq S-1.$$

Constructed as the Laplace transform with respect to the first variable of certain adequate functions $W_j(\tau,\epsilon)$, $0 \leq j \leq S-1$.

Strategy followed - Step 1

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Let $r := \frac{r_2}{s_2 k}$.

We fix $\epsilon \in \mathcal{E}$ and consider the problem

$$((T^{k+1}\partial_T)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(T^{k+1}\partial_T)^{s_1}+a_1)\partial_z^S Y(T,z,\epsilon)$$

$$= \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}T^s(\partial_T^{\kappa_0}\partial_z^{\kappa_1}Y)(T,z,\epsilon),$$

$$(\partial_z^j Y)(T,0,\epsilon) = Y_j(T,\epsilon), \qquad 0 \le j \le S-1.$$

Strategy followed - Step 1

$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2} + a_2)(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1} + a_1)\partial_z^S X(t,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon)$$

$$Y(t,z,\epsilon) = X(\epsilon^{-r}t,z,\epsilon)$$

$$((T^{k+1}\partial_T)^{s_2} + a_2)(\epsilon^{r_1-s_1rk}(T^{k+1}\partial_T)^{s_1} + a_1)\partial_z^S Y(T,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}T^s(\partial_T^{\kappa_0}\partial_z^{\kappa_1}Y)(T,z,\epsilon)$$

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$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2}+a_2)(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1)\partial_z^S X(t,z,\epsilon)$$
$$=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon)$$

$$Y(t,z,\epsilon)=X(\epsilon^{-r}t,z,\epsilon)$$

$$((T^{k+1}\partial_T)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(T^{k+1}\partial_T)^{s_1}+a_1)\partial_z^S Y(T,z,\epsilon)$$
$$=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}T^s(\partial_T^{\kappa_0}\partial_z^{\kappa_1}Y)(T,z,\epsilon)$$

- The perturbation related to one of the singular irregular operators disappears.
- The domain of definition for the solution might depend on ϵ.
- The coefficients in the equation might possess a pole at 0.

Strategy followed - Step 2

We take into account the property

$$T^{\delta_{\kappa_0}}T^{\kappa_0(k+1)}\partial_T^{\kappa_0}=T^{\delta_{\kappa_0}}\left((T^{k+1}\partial_T)^{\kappa_0}+\sum_{1\le p\le \kappa_0-1}A_{\kappa_0,p}T^{k(\kappa_0-p)}(T^{k+1}\partial_T)^p\right),$$

H. Tahara, H. Yamazawa, Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations, Journal of Differential Equations, Volume 255, Issue 10, 15 November 2013, 3592–3637.

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We take into account the property

$$T^{\delta_{\kappa_0}} T^{\kappa_0(k+1)} \partial_T^{\kappa_0} = T^{\delta_{\kappa_0}} \left((T^{k+1}\partial_T)^{\kappa_0} + \sum_{1 \le p \le \kappa_0 - 1} A_{\kappa_0,p} T^{k(\kappa_0-p)} (T^{k+1}\partial_T)^p \right),$$

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H. Tahara, H. Yamazawa, Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations, Journal of Differential Equations, Volume 255, Issue 10, 15 November 2013, 3592–3637.

and the properties of a modified formal Borel transform

$$\mathcal{B}_{m_k}(T^{k+1}\partial_T \hat{f}(T))(\tau) = k\tau^k \mathcal{B}_{m_k}(\hat{f}(T))(\tau),$$

$$\mathcal{B}_{m_k}(T^m \hat{f}(T))(\tau) = \frac{\tau^k}{\Gamma\left(\frac{m}{k}\right)} \int_0^{\tau^k} (\tau^k - s)^{\frac{m}{k}-1} \mathcal{B}_{m_k}(\hat{f}(T))(s^{1/k}) \frac{ds}{s}.$$

Strategy followed - Step 2

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$$((T^{k+1}\partial_T)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(T^{k+1}\partial_T)^{s_1}+a_1)\partial_z^S Y(T,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}T^s(\partial_T^{\kappa_0}\partial_z^{\kappa_1}Y)(T,z,\epsilon)$$

$$((k\tau^k)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(k\tau^k)^{s_1}+a_1)\partial_z^S W(\tau,z,\epsilon)$$

$$= \sum_{\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\left[\frac{\tau^k k^{\kappa_0}}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}s^{\kappa_0}\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right.$$

$$\left.+\sum_{p=1}^{\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k k^p}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}s^p\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right]$$

Strategy followed - Step 2

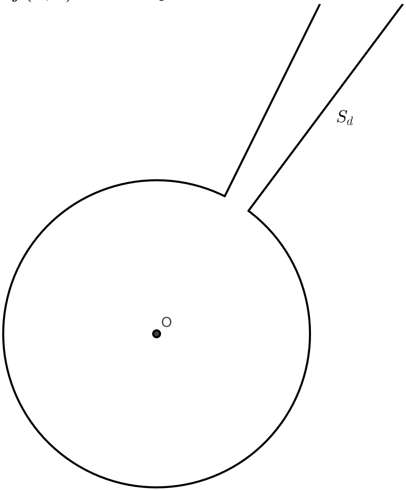
$$((k\tau^k)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(k\tau^k)^{s_1}+a_1)\partial_z^S W(\tau,z,\epsilon)$$
$$=\sum_{\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\left[\frac{\tau^k k^{\kappa_0}}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}s^{\kappa_0}\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right.$$
$$\left.+\sum_{p=1}^{\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k k^p}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}s^p\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right]$$

We construct the formal solution of this auxiliary problem in the form

$$W(\tau,z,\epsilon)=\sum_{\beta\geq 0}W_\beta(\tau,\epsilon)\frac{z^\beta}{\beta!}.$$

Initial conditions:

$$\tau \mapsto W_j(\tau, \epsilon), \quad 0 \leq j \leq S - 1$$



Strategy followed - Step 2

$$((k\tau^k)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(k\tau^k)^{s_1}+a_1)\partial_z^S W(\tau,z,\epsilon)$$
$$=\sum_{\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\left[\frac{\tau^k k^{\kappa_0}}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}s^{\kappa_0}\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right.$$
$$\left.+\sum_{p=1}^{\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k k^p}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}s^p\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right]$$

$$W(\tau,z,\epsilon)=\sum_{\beta\geq 0}W_\beta(\tau,\epsilon)\frac{z^\beta}{\beta!}$$

Strategy followed - Step 2

$$((k\tau^{k})^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(k\tau^{k})^{s_1}+a_1)\partial_z^S W(\tau,z,\epsilon)$$

$$=\sum_{\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\left[\frac{\tau^k k^{\kappa_0}}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}s^{\kappa_0}\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right.$$

$$\left.+\sum_{p=1}^{\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k k^p}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}s^p\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right]$$

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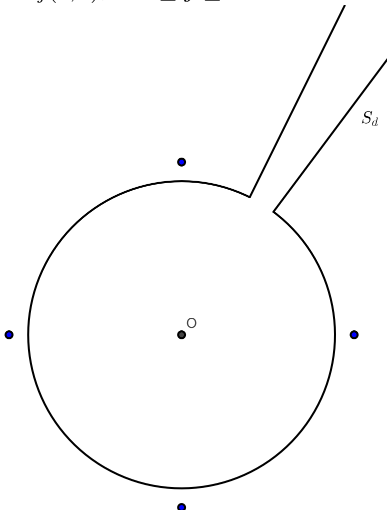


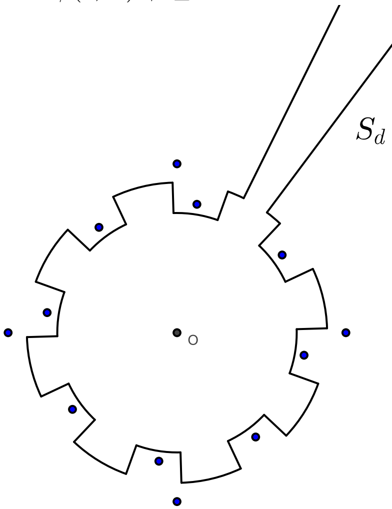
Strategy followed - Step 2

$$((k\tau^k)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(k\tau^k)^{s_1}+a_1)\partial_z^S W(\tau,z,\epsilon)$$
$$=\sum_{\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\left[\frac{\tau^k k^{\kappa_0}}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}s^{\kappa_0}\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right.$$
$$\left.+\sum_{p=1}^{\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k k^p}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}s^p\partial_z^{\kappa_1}W(s^{1/k},z,\epsilon)\frac{ds}{s}\right]$$

$$W(\tau,z,\epsilon)=\sum_{\beta\geq 0}W_\beta(\tau,\epsilon)\frac{z^\beta}{\beta!}$$

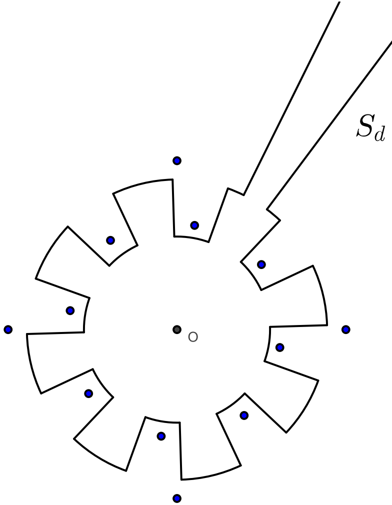
Given $\epsilon_1 \in \mathcal{E}$,

$$\tau \mapsto W_\beta(\tau, \epsilon_1),\ \beta \geq S$$

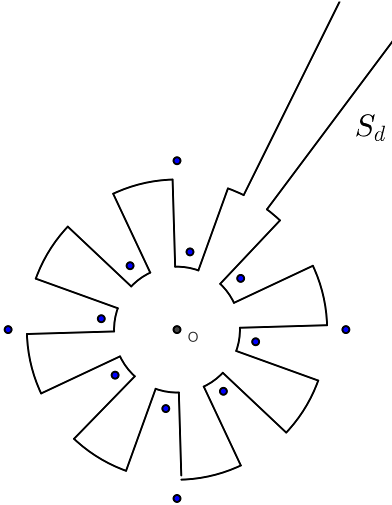


Given $\epsilon_2 \in \mathcal{E}$, $|\epsilon_2| < |\epsilon_1|$,

$$\tau \mapsto W_\beta(\tau, \epsilon_2),\ \beta \geq S$$



Given $\epsilon_3 \in \mathcal{E}$, $|\epsilon_3| < |\epsilon_2| < |\epsilon_1|$,
$\tau \mapsto W_\beta(\tau, \epsilon_3)$, $\beta \geq S$



Strategy followed - Step 3

The coefficients of

$$W(\tau,\epsilon,z)=\sum_{\beta\geq 0} W_{\beta}(\tau,\epsilon)\frac{z^{\beta}}{\beta!}$$

rest in an appropriate Banach space of functions (which depends on $\epsilon \in \mathcal{E}$), chosen to:

- avoid the singularities of both nature.
- preserve the exponential growth along direction d.
- behave appropriately with respect to certain operators.

Strategy followed - Step 3

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- avoid the singularities of both nature.
- preserve the exponential growth along direction d.
- behave appropriately with respect to certain operators.

One defines

$$Y(T,z,\epsilon):=\sum_{\beta\geq 0}\mathcal{L}_{k}^{d}(W_{\beta}(\tau,\epsilon))(T)\frac{z^{\beta}}{\beta!}.$$

Strategy followed - Step 3

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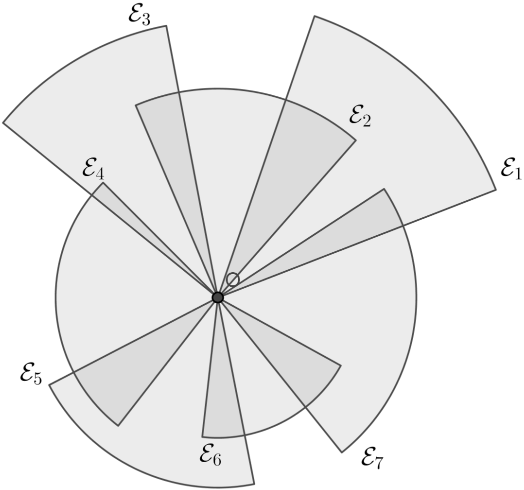
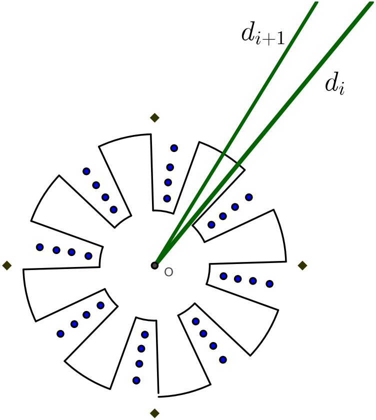
$$Y(T,z,\epsilon) := \sum_{\beta\geq 0} \mathcal{L}_k^d(W_\beta(\tau,\epsilon))(T)\frac{z^\beta}{\beta!}.$$

The function $X(t,z,\epsilon) = Y(\epsilon^r t, z, \epsilon)$ is holomorphic in $\mathcal{T} \times D \times \mathcal{E}$ and provides a solution of the main problem.



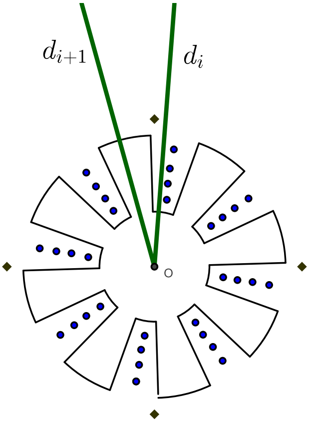
Let $(\mathcal{E}_i)_{1\leq i\leq \nu}$
be a good covering.

Formal solution and asymptotics



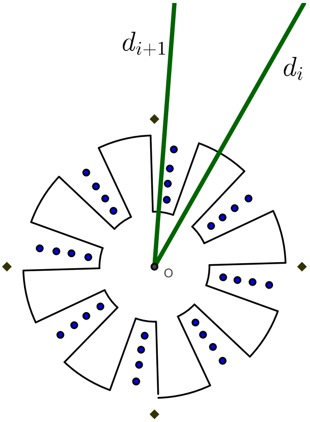
Then, one can choose $d_i \equiv d_{i+1}$, and reformulate the problem for a new good covering.

Formal solution and asymptotics



$$\sup_{t\in\mathcal{T},z\in D}|X_{i+1}(t,z,\epsilon)-X_i(t,z,\epsilon)|\le K\exp\left(-\frac{M}{|\epsilon|^{r_2/s_2}}\right),$$

Formal solution and asymptotics



$$\sup_{t \in \mathcal{T}, z \in D} |X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \leq K \exp\left(-\frac{M}{|\epsilon|^{r_1/s_1}}\right),$$

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For every $1 \leq i \leq \nu$,

$$X_i(t,z,\epsilon) = a(t,z,\epsilon) + X_i^1(t,z,\epsilon) + X_i^2(t,z,\epsilon),$$

$\epsilon \mapsto X_i^1(t,z,\epsilon)$ is a $\mathbb{E}$-valued function which admits $\hat{X}^1(t,z,\epsilon)$ as its r_1/s_1-Gevrey asymptotic expansion on $\mathcal{E}_i$,

$\epsilon \mapsto X_i^2(t,z,\epsilon)$ is a $\mathbb{E}$-valued function which admits $\hat{X}^2(t,z,\epsilon)$ as its r_2/s_2-Gevrey asymptotic expansion on $\mathcal{E}_i$.

Multisummability

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We also provide a condition concerning the opening of $\mathcal{E}_i$ and the distribution of singularities so that
$\epsilon \mapsto X_i^1(t,z,\epsilon)$ is the r_1/s_1-sum of $\hat{X}^2(t,z,\epsilon)$ on $\mathcal{E}_i$, and
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$\hat{X}(t,z,\epsilon)$ is $(r_2/s_2, r_1/s_1)-$summable on $\mathcal{E}_i$ and its $(r_2/s_2, r_1/s_1)-$sum is $X_i(t,z,\epsilon)$ on $\mathcal{E}_i$, using the characterization of multisummability in

W. Balser, From divergent power series to analytic functions. Theory and application of multisummable power series. Lecture Notes in Mathematics, 1582. Springer-Velag, Berlin, 1994, x+108 pp.