Holomorphic and Formal Gevrey Solutions of a Singular Integro-Differential Equation

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Analytic, Algebraic and Geometric Aspects of Differential Equations

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Outline of Presentation

1. Background and Main Result
   - Bielawski's system
   - Main Result
2. Preliminaries
   - The function φ
   - The function Φ_s
3. Proof of Main Result
   - Rewriting the equation
   - Fixed-point formulation
   - Key estimates

Outline

— Motivation —

To construct a Ricci-flat Kähler metric having certain properties, Bielawski studied the system

$$(e) \qquad \begin{cases} t\partial_t u = a(x)\,t + \rho(x)\,u + R_2\Big(t, x, u, \partial_x u, w_1, \dots, w_N\Big) \\ \partial_t w_j = \sum_{|\gamma|\le 2} c^j_\gamma(x)\partial^\gamma_x u + e_j(x) \quad (j = 1, 2, ., N), \end{cases}$$

where $R_2(t, x, u, v, w) = \sum_{p+q+\alpha+2|\beta|\ge 2} a_{p,q,\alpha,\beta}(x) t^p u^q v^\alpha w^\beta$.

- Without the presence of the w_j's, we get the equation of Gérard and Tahara
- Bielawski showed that if $\rho(0) \notin \mathbb{N}^*$, then (e) has a unique holomorphic solution satisfying $u(0, x) \equiv 0$ and $w_j(0, x) \equiv 0$ $(j = 1, 2, \dots, N)$.

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— The Equation —

Let $(t,x) \in \mathbb{C}_t \times \mathbb{C}_x^n$, m and q be positive integers and $d \geq 1$. We consider the equation

$$(t\partial_t)^m u = F\left(t,x,\left\{\partial_t^{-a}(t\partial_t)^b \partial_x^\alpha u\right\}_{(a,b,\alpha)\in\Lambda}\right), \tag{1}$$

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$$\Lambda = \left\{(a, b, \alpha) \in \mathbb{N}^{2+n};\ a \leq q,\ b + \lceil d\,|\alpha| \rceil - a \leq m \text{ and } b \leq m\right\}.$$

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We assume that $F(t,x,\{Y_{a,b,\alpha}\})$ may be expanded as

$$F(t,x,\{Y_{a,b,\alpha}\}) = g_{1,0}(x)t + \sum_{i=0}^{m-1} \lambda_i(x) Y_{0,i,0} + \sum_{\substack{(a,b,\alpha)\in\Lambda \\ a\neq 0}} f_{a,b,\alpha}(x) Y_{a,b,\alpha} + G(t,x,Y),$$

where $G(t,x,\{Y_{a,b,\alpha}\})$ consists of terms that are of degree at least 2 in t and Y, that is,

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Thus, our equation takes the form

$$\left((t\partial_t)^m - \sum_{i=0}^{m-1} \lambda_i(x)\,(t\partial_t)^i\right)u = g_{1,0}(x)t + \sum_{\substack{(a,b,\alpha)\in\Lambda \\ a\neq 0}} f_{a,b,\alpha}(x)\mathcal{D}_{a,b,\alpha}u$$
$$+ \sum_{p+|\nu|\geq 2} g_{p,\nu}(x)t^p \prod_{(a,b,\alpha)\in\Lambda} \left(\mathcal{D}_{a,b,\alpha}u\right)^{\nu_{a,b,\alpha}},$$

where we denoted by $\mathcal{D}_{a,b,\alpha}$ the operator $\partial_t^{-a}\,(t\partial_t)^b\,\partial_x^\alpha$.

— Notations —

$\mathbb{C}[[t,x]]$: ring of formal power series in (t,x)

$\mathbb{C}\{t,x\}$: subring of convergent power series in (t,x)

$\mathbb{C}\{t,x\}_{(\sigma,s)}$: formal Gevrey class of order (σ,s); all power series $u=\sum_{i,j\geq 0} u_{i,j}t^i x^j$ such that $\sum_{i,j\geq 0} \frac{u_{i,j}}{i!^{\sigma-1}j!^{s-1}} t^i x^j$ is in $\mathbb{C}\{t,x\}$

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— Main Result —

$$\left((t\partial_t)^m - \sum_{i=0}^{m-1} \lambda_i(x)\,(t\partial_t)^i\right)u = g_{1,0}(x)t + \sum_{\substack{(a,b,\alpha)\in\Lambda \\ a\neq 0}} f_{a,b,\alpha}(x)\mathcal{D}_{a,b,\alpha}u$$
$$+ \sum_{p+|\nu|\geq 2} g_{p,\nu}(x)t^p \prod_{(a,b,\alpha)\in\Lambda} \left(\mathcal{D}_{a,b,\alpha}u\right)^{\nu_{a,b,\alpha}} \quad (2)$$

Theorem (Main Result)

Suppose that the coefficients $\lambda_i(x)$, $g_{1,0}(x)$, $f_{a,b,\alpha}(x)$ and $g_{p,\nu}(x)$ are in $\mathbb{C}\{x\}_d$. If $k^m - \sum_{i=0}^{m-1}\lambda_i(0)k^i \neq 0$ for all $k \in \mathbb{N}^*$, then (2) has a unique formal series solution $u = \sum_{k=1}^{\infty} u_k(x)t^k$ in $\mathbb{C}^*\{t,x\}_{(1,d)}$.

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— Definitions —

- Given two formal power series $a(t,x) = \sum a_{i,\alpha} t^i x^\alpha$ and $A(t,x) = \sum A_{i,\alpha} t^i x^\alpha$, we say that

  $A(t,x)$ majorizes $a(t,x)$ if and only if $|a_{i,\alpha}| \leq A_{i,\alpha}$.

- We write $a(t,x) \ll A(t,x)$ to denote this fact.
- We define

$$\varphi(z) = \frac{1}{4S} \sum_{k=0}^{\infty} \frac{z^k}{(k+1)^2},$$

where $S = \frac{1}{6}\pi^2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$.

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— Important properties —

$$\varphi(z) = \frac{1}{4S} \sum_{k=0}^{\infty} \frac{z^k}{(k+1)^2}.$$

1. $\varphi(z)\,\varphi(z) \ll \varphi(z)$
2. For any $\varepsilon \in (0,1)$, there exists a constant $C_\varepsilon > 0$ such that for all $p \in \mathbb{N}$,

$$\frac{1}{1-\varepsilon z} D^p \varphi(z) \ll C_\varepsilon D^p \varphi(z).$$

3. If $R \in (0, \frac{1}{4})$, then for any $p, q \in \mathbb{N}^*$, we have

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— Another majorant function —

For $s \in \mathbb{N}^*$, we define (cf. Pongérard-Wagschal, 2007):

$$\Phi_s(t,x) = \sum_{p=0}^{\infty} t^p \frac{D^{sp}\varphi(x)}{(sp)!}.$$

- The series $\Phi_s(t, x/R)$ converges for all $|t|^{1/s} + |x| < R$.
- $\Phi_s(t,x)\,\Phi_s(t,x) \ll \Phi_s(t,x)$
- For any $0 < \varepsilon, \delta < 1$, there exists $B_{\varepsilon,\delta} > 0$ such that

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— Notation —

For any $\phi(t,x) = \sum_{i=0}^{\infty} \phi_i(t)x^i$, we define

$$\phi^{(d)}(t,x) = \sum_{i=0}^{\infty} \phi_i(t)(i!)^{d-1}x^i.$$

- $\phi \ll \psi$ if and only if $\phi^{(d)} \ll \psi^{(d)}$
- $\phi^{(d)}\psi^{(d)} \ll (\phi\psi)^{(d)}$
- If $u \in \mathbb{C}\{t,x\}_{(1,d)}$ then we there exists a holomorphic function $\hat{u} \in \mathbb{C}\{t,x\}$ such that $\hat{u}^{(d)} = u$.

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— Characteristic polynomial —

Define the polynomial

$$P(k) = k^m - \sum_{i=0}^{m-1} \lambda_i(0) k^i.$$

- The operator $P(t\partial_t)$ maps $\mathbb{C}^*[[t,x]]$ into itself.
- In fact, if $P(k) \neq 0$ for all $k \in \mathbb{N}^*$, then $P(t\partial_t)$ is a bijection on $\mathbb{C}^*[[t,x]]$. If $w(t,x) = \sum_{k=1}^{\infty} w_k(x) t^k$, then

$$P(t\partial_t)^{-1} w = \sum_{k=1}^{\infty} \frac{w_k(x) t^k}{P(k)}.$$

- Since $P(k) \neq 0$ for all $k \in \mathbb{N}^*$, there exists $A > 0$ such that $k^m \leq A\,|P(k)|$ for all k.

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- Since $P(k) \neq 0$ for all $k \in \mathbb{N}^*$, there exists $A > 0$ such that $k^m \leq A\,|P(k)|$ for all k.

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Define the polynomial

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$$\left((t\partial_t)^m - \sum_{i=0}^{m-1} \lambda_i(x)\,(t\partial_t)^i\right)u = g_{1,0}(x)t + \sum_{\substack{(a,b,\alpha)\in\Lambda \\ a\neq 0}} f_{a,b,\alpha}(x)\mathcal{D}_{a,b,\alpha}u$$

$$+ \sum_{p+|\nu|\geq 2} g_{p,\nu}(x)t^p \prod_{(a,b,\alpha)\in\Lambda} \left(\mathcal{D}_{a,b,\alpha}u\right)^{\nu_{a,b,\alpha}},$$

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— Equivalent equation —

Our equation becomes

$$v = \hat{g}_{1,0}^{(d)}(x)t + \sum_{i=0}^{m-1} \hat{\lambda}_i^{*(d)}(x)\,(t\partial_t)^i\,P\,(t\partial_t)^{-1}\,v$$
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Here, $\hat{g}_{1,0}$, $\hat{\lambda}_i^*$, $\hat{f}_{a,b,\alpha}$ and $\hat{g}_{p,\nu}$ are assumed to be holomorphic in some neighborhood of $\{|x| \leq R_0\}$ and bounded there by some $M > 0$.

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— Definition of mapping —

We then define the map T as follows:

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We shall show that this is a strict contractive mapping of a closed ball in some Banach space into itself.

— The Banach space —

We fix positive constants $r \in (0, r_0)$, $R \in (0, R_0)$ and $c \in (0, 1)$, and set $s = 2(m+q)$. We then define the Banach space

$$\mathcal{B} = \left\{ u \in \mathbb{C}^*[[t,x]] : \exists L > 0 \text{ such that } u \ll Lt\Phi_s^{(d)}\left(\frac{t}{cr}, \frac{|x|}{R}\right) \right\}$$

with norm

$$\|u\| = \inf\left\{ L > 0 : \, u \ll Lt\Phi_s^{(d)}\left(\frac{t}{cr}, \frac{|x|}{R}\right) \right\}.$$

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— Key lemmas —

Lemma (1)

There exists a constant $B_d > 0$ *such that for all* p *and* j *in* $\mathbb{N}$, *we have*

$$D^p\left(D^j\varphi\left(\tfrac{z}{R}\right)\right)^{(d)} \ll B_d\, R^{\lceil dp\rceil - p}\left(D^{j+\lceil dp\rceil}\varphi\left(\tfrac{z}{R}\right)\right)^{(d)}.$$

Note: We may take $B_d = \lceil d \rceil$. This is trivial when $d = 1$.

— Key lemmas —

Lemma (2)

There exists a constant $C > 0$ *such that for all* $v, v' \in \mathcal{B}$ *and* $(a, b, \alpha), (a', b', \alpha') \in \Lambda$, *we have*

1. $\mathcal{D}_{a,b,\alpha} P(t\partial_t)^{-1} v \ll C \|v\| \, c\Phi_s^{(d)} \left(\frac{t}{cr}, \frac{|x|}{R} \right)$
2. $$\left(\mathcal{D}_{a,b,\alpha} P(t\partial_t)^{-1} v\right)\left(\mathcal{D}_{a',b',\alpha'} P(t\partial_t)^{-1} v'\right) \ll C \|v\| \, \|v'\| \, ct\Phi_s^{(d)} \left(\frac{t}{cr}, \frac{|x|}{R} \right)$$

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$$\ll C \|v\| \, \|v'\| \, ct\Phi_s^{(d)}\left(\frac{t}{cr}, \frac{|x|}{R}\right)$$

— Proof of (1) —

Since $v \in \mathcal{B}$, we have $v(0,x) \ll \|v\|\, t\Phi_s^{(d)}(0, \frac{|x|}{R})$. Thus

$$\mathcal{D}_{a,b,\alpha} P(t\partial_t)^{-1} v = \sum_{k=1}^{\infty} \frac{k^b}{P(k)} \frac{k!}{(k+a)!} \frac{\partial_x^\alpha \partial_t^k v(0,x)}{k!} t^{k+a}$$

$$\ll B_d R^{\lceil d|\alpha| \rceil - |\alpha|} \|v\| \sum_{k=1}^{\infty} C_k (cr)^{1-k} \frac{(D^{sk-s+\lceil d|\alpha| \rceil} \varphi(\frac{|x|}{R}))^{(d)}}{(sk-s+\lceil d\,|\alpha| \rceil)!} t^{k+a},$$

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— Recall: —

The map T was defined as follows:

$$T(v) = \hat{g}^{(d)}_{1,0}(x)t + \sum_{i=0}^{m-1} \hat{\lambda}^{*(d)}_i(x) \left(t\partial_t\right)^i P\left(t\partial_t\right)^{-1} v$$
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— The linear part of $T(v) - T(v')$ —

The following hold:

$$\sum_{i=0}^{m-1} \hat{\lambda}_i^{*(d)}(x)(t\partial_t)^i P(t\partial_t)^{-1}(v - v') \ll 4RK_1A \left\|v - v'\right\| t\Phi_s^{(d)}$$

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— The nonlinear part —

Denote the third summation in (4) by $\hat{G}(t,x,Yv)$, where $Yv = \{Y_\gamma v\}$ with $\gamma = (a,b,\alpha) \in \Lambda$. As noted by Pongérard, the difference may be written as

$$\hat{G}(t,x,Yv) - \hat{G}(t,x,Yv') = \sum_{p\geq 1,\gamma\in\Lambda} \hat{g}^{(d)}_{p,\nu}(x) t^p Y_\gamma (v - v') + \sum_{\gamma,\delta\in\Lambda} \Big(h^{(d)}_{\gamma,\delta}(t,x,Yv,Yv')\, Y_\gamma v\, Y_\delta (v-v') - k^{(d)}_{\gamma,\delta}(t,x,Yv,Yv')\, Y_\delta v'\, Y_\gamma (v-v') \Big).$$

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Since $h_{\gamma,\delta}$ and $k_{\gamma,\delta}$ are holomorphic wrt all their arguments, we may show that

$$h_{\gamma,\delta}^{(d)} \text{ and } k_{\gamma,\delta}^{(d)} \ll 4MB_\epsilon \Phi_s^{(d)}$$

— Resulting estimates —

We will eventually obtain the estimates

$$\left\|T(v)-T(v')\right\| \leq \left(L_1 R + L_2 c\right)\left\|v-v'\right\|$$

and

$$\left\|T(v)\right\| \leq b\left(L_1 R + L_2 c\right) + L_3$$

for some constants L_1, L_2 and L_3.

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— Conclusion —

Thus, there exists a unique $v \in B(b)$ satisfying (3).

As

$$v(t,x) \ll bt\Phi_s^{(d)}\left(\frac{t}{cr}, \frac{x_1 + \cdots + x_n}{R}\right),$$

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— An invitation —

International Conference on Partial Differential Equations:

General Theory and Variational Problems

January 11–15, 2016

Costabella Tropical Beach Hotel

Mactan Island, Cebu, Philippines

Maraming salamat po!

(Thank you very much!)