Mean values and real analytic functions

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The Pizzetti mean-value formula states that the integral mean of a smooth function u over the Euclidean sphere of radius r is expressed as a series of even powers of r whose coefficients are given by iterated Laplacian evaluated on u at the center of the sphere multiplied by numerical factors. In the case of 2-dimensional sphere the formula was derived already in 1909 by Pizzetti [22].

Its extensions to the case of means over Euclidean spheres and balls in arbitrary dimension for polyharmonic functions and the inverse mean-value properties were derived by Nicolesco [20]. Later on the Pizzetti serii were studied from different points of view, see [12, 5, 8] and references therein.

In particular, it was proved that real analytic functions can be characterized by convergence of the Pizzetti serii [3, 15]. The Pizzetti formulas were also extended to the case of means on Riemannian manifolds [13] and on the Heisenberg group [6, 21], and to the Dunkl-Laplace operator on $\mathbb{R}^n$ [16, 23]. A generalized mean value theorem with respect to a general Borel measure supported by the unit real ball for solutions of a system of homogeneous partial differential equations was derived by Zalcman [27].

In the lecture we introduce integral mean value functions which are averages of integral means over spheres/ balls and over their images under the action of a discrete group of complex rotations. In the case of real analytic functions we derive higher order Pizzetti's formulas.

Applications:

- a maximum principle for polyharmonic functions,
- a characterization of functions of Laplacian growth,

• a characterization of convergent solutions to the initial value problem for the higher order heat type equation $\partial_t u = \Delta^l u$, $u(0, \cdot) = \varphi$ where $l \in \mathbb{N}$ and φ is real analytic.

In the second part of the lecture we shall give a characterization of real analytic functions in terms of integral means. The characterization justifies a definition of analytic functions on metric measure spaces.

1 Higher order mean value functions

Let $k \in \mathbb{N}$. Denote by ϵ the transformation of $\mathbb{C}^n$ into $\mathbb{C}^n$ given by

$$\epsilon(z_1, \ldots, z_n) = (e^{2\pi\imath/k} z_1, \ldots, e^{2\pi\imath/k} z_n).$$

Let u be a continuous function defined on a complex neighborhood U of an open set $\Omega \subset \mathbb{R}^n$.
For $x \in \Omega$ and $0 < r < \mathrm{dist}(x, \partial U)$ we define the *spherical* and *solid mean value functions of order k.*

$$N_k(u;\, x, r) = \frac{1}{kn\sigma(n)} \sum_{j=0}^{k-1} \int\limits_{S^{n-1}(0,1)} u\big(x + r\epsilon^j(y)\big)\, dS(y), \quad (1.1a)$$

$$M_k(u;\, x, r) = \frac{1}{k\sigma(n)} \sum_{j=0}^{k-1} \int_{B^n(0,1)} u\big(x + r\epsilon^j(y)\big)\, dy, \quad (1.1b)$$

where $\sigma(n) = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball $B^n(0,1)$ in $\mathbb{R}^n$ and $dS = dS^{n-1}$ is the natural measure on the unit sphere $S(0,1) = S^{n-1}(0,1)$.

Note that if k is odd, then
$N_k(u;\, x, r) = N_{2k}(u;\, x, r)$ and
$M_k(u;\, x, r) = M_{2k}(u;\, x, r)$.
In particular

$$N_1(u;\, x, r) = N_2(u;\, x, r) = \frac{1}{n\sigma(n)} \int_{S(0,1)} u(x + ry)\, dS(y)$$

and

$$M_1(u;\, x, r) = M_2(u;\, x, r) = \frac{1}{\sigma(n)} \int_{B(0,1)} u\big(x + ry\big)\, dy.$$

2 Higher order Pizzetti's formulas

In this section we assume that $u \in \mathcal{A}(\Omega)$ is a real analytic function on an open set $\Omega \subset \mathbb{R}^n$. Then u extends to a function $\widetilde{u}$ holomorphic on a complex neighborhood U of Ω and for any $x \in \Omega$ it holds

$$\widetilde{u}(y) = \sum\nolimits_{\kappa \in \mathbb{N}_0^n} \frac{1}{\kappa!} \frac{\partial^{|\kappa|} u}{\partial x^\kappa}(x)(y-x)^\kappa \tag{2.1}$$

for $\|y - x\| < \rho(x)$ with some $\rho \in C^0(\Omega, \mathbb{R}_+)$.

Hence the functions N_k and M_k are well defined for $x \in \Omega$ and r small enough.

Theorem 2.1 (Higher order Pizzetti's formulas.) *Let $k = 2l$ with $l \in \mathbb{N}$, $u \in \mathcal{A}(\Omega)$ and $x \in \Omega$. Then $N_k(u; x, r)$ and $M_k(u; x, r)$ are real analytic functions* *at the origin and for* r *small enough it holds*

$$N_k(u;\, x, r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2}\right)_{lm} (lm)!}\, r^{2lm}, \tag{2.2a}$$

$$M_k(u;\, x, r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2} + 1\right)_{lm} (lm)!}\, r^{2lm}, \tag{2.2b}$$

where Δ *is the Laplace operator and* $(a)_m = a(a+1)\cdots(a+m-1)$ *for* $m \in \mathbb{N}$ *is the Pochhammer symbol.*

The proof is done by expending u into Taylor power series, noting that the integral of y^κ over $B(0,1)$ vanishes if at least one of the coordinates $\kappa_\imath$ of κ is odd, using the following property of of the roots of unity

$$\sum_{j=0}^{k} e^{2j|\kappa|\pi\imath/k} = \begin{cases} k & \text{if } |\kappa| = km \text{ for some } m \in \mathbb{N}_0, \\ 0 & \text{otherwise,} \end{cases}$$

and the formula [9, formula 676, 11], and finally recognizing in the obtained expression the powers of the laplacian multiplied by numerical factors. $\square$

Passing to the limit $k \to \infty$ in the formulas (2.2a) and (2.2b) we get the following mean value formulas for real analytic functions.

Corollary 1 *Let* $u \in \mathcal{A}(\Omega)$ *and let* $\widetilde{u}$ *be a holomorphic extension of* u*. Then for* $x \in \Omega$ *and* $r > 0$ *small enough it holds*

$$u(x) = \frac{1}{2n\sigma(n)\pi} \int_{S^1_\zeta(0,1)} \int_{S^{n-1}(0,1)} \widetilde{u}(x + r\zeta y)\, dS^{n-1}(y) \cdot dS^1(\zeta)$$

$$= \frac{1}{2\sigma(n)\pi} \int_{S^1_\zeta(0,1)} \int_{B^n(0,1)} \widetilde{u}(x + r\zeta y)\, dB^n(y) \cdot dS^1(\zeta),$$

where $S^1_\zeta(0,1) = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Real analytic functions can be characterized as those smooth ones for which the Pizzetti's serii converge.

Theorem 2.2 *Let* $l \in \mathbb{N}$, $\rho \in C^0(\Omega;\mathbb{R}_+)$ *and* $u \in C^\infty(\Omega)$. *If the series*

$$\widetilde{N}(x,r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!}\, r^{2lm}$$

is convergent locally uniformly in $\{(x,r) : x \in \Omega, |r| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$ *and* $N_{2l}(u;\, x,r) = \widetilde{N}(x,r)$ *for* $x \in \Omega$ *and* $0 < r < \min\left(\rho(x), \operatorname{dist}(x,\partial\Omega)\right)$.

Corollary 2 *Under the assumptions of Theorem 2.2 if the series*

$$\widetilde{M}(x,r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2}+1\right)_{lm} (lm)!} r^{2lm}$$

is convergent locally uniformly in $\{(x,r) : x \in \Omega, |r| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$ *and* $M_{2l}(u;\, x, r) = \widetilde{M}(x,r)$ *for* $x \in \Omega$ *and* $0 < r < \min\big(\rho(x), \operatorname{dist}(x, \partial\Omega)\big)$.

3 Maximum principle for polyharmonic functions

It is well known that modulus of a function u harmonic on a connected domain $\Omega \subset \mathbb{R}^n$ can not attain its maximum at an interior point of Ω unless u is constant. On the other hand this maximum principle does not extends to l-polyharmonic functions, i.e., solutions to $\Delta^l u = 0$ with $l \geq 2$. However due real analyticity of such functions by the formula (2.2b) we obtain the following maximum principle for polyharmonic functions.

Theorem 3.1 *Let u be a real valued, l-polyharmonic function on a connected open set $\Omega \subset \mathbb{R}^n$, $l \in \mathbb{N}$. Denote by $\widetilde{u}$ its holomorphic extension to a connected complex neighborhood U of Ω. If for some $x_0 \in \Omega$ and $r_0 > 0$ it holds*

$$\pm u(x_0) \geq \pm \mathrm{Re}\, \widetilde{u}(y) \quad \textit{for} \quad y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j\big(B(0, r_0)\big),$$

where $\epsilon(z) = e^{\pi \imath / l} z$, then u is constant on Ω.

Proof. Since u is l-polyharmonic the series in (2.2b) terminates at the first term. Hence

$$M_{2l}(u;\, x, r) = \frac{1}{l\sigma(n)} \sum_{j=0}^{l-1} \int\limits_{B^n(0,r)} \widetilde{u}\big(x+\epsilon^j(y)\big)\,dy = u(x).$$

So $M_{2l}(u;\, x_0, r) = u(x_0)$ for $0 < r < \rho(x_0)$ and the assumption implies that $\operatorname{Re} \widetilde{u}(y) = u(x_0)$ for $y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j\big(B(0, r_1)\big)$ with $0 < r_1 < \min(r_0, \rho(x_0))$.
It follows that u is constant on Ω. $\square$

Corollary 3 *Let u be a real valued, l-polyharmonic function on a connected open set $\Omega \subset \mathbb{R}^n$, $l \in \mathbb{N}$. Denote by $\widetilde{u}$ its holomorphic extension to a connected complex neighborhood U of Ω. If for some $x_0 \in \Omega$ and $r_0 > 0$ it holds*

$$|u(x_0)| \geq |\widetilde{u}(y)| \quad \textit{for} \quad y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j \big(B(0, r_0)\big),$$

then u is constant on Ω.

4 Functions of l-Laplacian growth

The notion of functions of Laplacian growth was introduced by Aronszajn et al. [1, Chapter II]. Here we introduce its following generalization.

Definition. Let $l \in \mathbb{N}$, $\varrho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^n$ is of *l-Laplacian growth* (ϱ, τ) *on* Ω if for every compact set $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that for any $m \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^{lm} u(x)| \leq C (2lm)!^{1-1/\varrho} (\tau + \varepsilon)^{2lm}. \quad (4.1)$$

It follows by [14, Theorem] that a function u of l-Laplacian growth (ϱ, τ) on Ω is real analytic on Ω. Hence (2.2a) and (2.2b) hold for any $x \in \Omega$ and r small enough. However due to the estimation (4.1) both functions N_{2l} and M_{2l} extend to entire functions of r.

Theorem 4.1 *Let $l \in \mathbb{N}$, $\varrho > 0$, $\tau \geq 0$ and $u \in C^\infty(\Omega)$. If u is of l-Laplacian growth (ϱ, τ) on Ω, then $N_{2l}(u; x, r)$ and $M_{2l}(u; x, r)$ as functions of r extend holomorphically to entire functions of exponential growth $(\varrho, \tau^\varrho/\varrho)$ locally uniformly in Ω.*

Theorem 4.2 *Let $l \in \mathbb{N}$, $\varrho > 0$, $\tau \geq 0$ and $u \in \mathcal{A}(\Omega)$. Assume that $M_{2l}(u; x, r)$ (resp. $N_{2l}(u; x, r)$) defined for $x \in \Omega$ and $0 \leq r < \operatorname{dist}(x, \partial\Omega)$ extends holomorphically to an entire function $\widetilde{M_{2l}}(u; x, z)$ (resp. $\widetilde{N}_{2l}(u; x, z)$) of exponential growth (ϱ, τ) locally uniformly in Ω.*
Then u is of l-Laplacian growth $\big(\varrho, (\tau\varrho)^{1/\varrho}\big)$.

5 Convergent solutions of higher order heat equations

For $l \in \mathbb{N}$ let us consider the initial value problem for the l-th order heat type equation

$$\begin{cases} \partial_t u - \Delta_x^l u &= 0, \\ u_{|t=0} &= u_0, \end{cases} \tag{5.1}$$

where $u_0 \in \mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^n$. Clearly, the unique formal power series solution of (5.1) is given by

$$\widehat{u}(t,x) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u_0(x)}{m!} t^m. \tag{5.2}$$

We ask when the solution u is an analytic function of the time variable at $t = 0$.

Theorem 5.1 *Let $0 < T \leq \infty$. The formal power series solution* (5.2) *of the initial value problem* (5.1) *is convergent for $|t| < T$ locally uniformly in Ω, iff $M_{2l}(u_0; x, r)$ and/or $N_{2l}(u_0; x, r)$ extend holomorphically to entire functions of exponential growth $\left(\frac{2l}{2l-1}, \frac{2l-1}{2l}(2lT)^{1-2l}\right)$ locally uniformly in Ω.*

Remark. The summability of the formal power series solution (5.2) of the equation (5.1) is studied by Michalik [19].

6 A characterization of real analyticity when $l = 1$

Theorem 6.1 (Mean-value property.) *Let $u \in \mathcal{A}(\Omega)$ and $x \in \Omega$. Then $M(u;\, x, R)$ and $N(u;\, x, R)$ are real analytic functions at the origin and for R small enough*

$$M(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}+1\right)_k k!} R^{2k},$$

$$N(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}.$$

Theorem 6.2 (Converse to the mean-value property.) *Let* $u \in C^\infty(\Omega)$ *and* $\rho \in C(\Omega, \mathbb{R}_+)$. *If*

$$\widetilde{M}(x,R) = \sum\nolimits_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}+1\right)_k k!} R^{2k}$$

or

$$\widetilde{N}(x,R) = \sum\nolimits_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}$$

is loc. uni. conv. in $\{(x,R) : x \in \Omega, |R| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$, $M = \widetilde{M}$ *and* $N = \widetilde{N}$.

Lemma 6.3 *Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^0(\Omega)$. Assume that there exist functions $v_{2l} \in C^0(\Omega)$ for $l \in \mathbb{N}_0$ and $\rho \in C^0(\Omega, \mathbb{R}_+)$ such that for any $x \in \Omega$ and $0 < R < \rho(x)$*

$$N(u;\, x, R) = \sum_{l=0}^{\infty} v_{2l}(x) R^{2l}. \tag{6.1}$$

Then $u \in C^\infty(\Omega)$ and $v_{2l} \in C^\infty(\Omega)$ for $l \in \mathbb{N}_0$.

Proof. Let $\widetilde{\eta}(r)$ be a smooth function on $[0,\infty)$ supported by $[0,1]$ with $n\sigma(n)\int_0^1 \widetilde{\eta}(r)r^{n-1}\,dr = 1$. Then

$$\eta^{\varepsilon}(y) = \frac{1}{\varepsilon^n}\widetilde{\eta}\left(\frac{|y|}{\varepsilon}\right)$$

is a radially symmetric mollifier supported by $\overline{B}(0,\varepsilon)$.

Integrating in the spherical coordinates we get

$$\int_{B(0,\varepsilon)} \eta^{\varepsilon}(y)\, dy = \int_0^1 n\sigma(n)\widetilde{\eta}(r)r^{n-1}\, dr = 1.$$

Since η^{ε} is radially symmetric we have

$$\begin{aligned} \Delta\eta^{\varepsilon}(y) &= \frac{1}{\varepsilon^{n+2}}\widetilde{\eta}''\left(\frac{|y|}{\varepsilon}\right) + \frac{1}{\varepsilon^{n+1}}\frac{n-1}{|y|}\widetilde{\eta}'\left(\frac{|y|}{\varepsilon}\right) \\ &:= L_{\varepsilon}(\widetilde{\eta})(|y|). \end{aligned} \tag{6.2}$$

For $x \in \Omega$ and $0 < \varepsilon < \rho(x)$ we compute

$$
\begin{aligned}
&\Delta\big(\eta^\varepsilon * u\big)(x) = \big(\Delta\eta^\varepsilon\big) * u(x) = \int_{B(0,\varepsilon)} \big(\Delta\eta^\varepsilon\big)(y)u(x-y)\,dy \\
&= \int_0^\varepsilon \bigg(\int_{S(0,1)} \big(\Delta\eta^\varepsilon\big)(rz)u(x-rz)\,dS(z)\bigg) r^{n-1}\,dr \\
&\overset{(6.2)}{=} \int_0^\varepsilon \int_{S(0,1)} u(x-rz)\,dS(z) L_\varepsilon(\widetilde{\eta})(r) r^{n-1}\,dr \\
&= \int_0^\varepsilon N(u;\, x, r) L_\varepsilon(\widetilde{\eta})(r) r^{n-1}\,dr \\
&\overset{(6.1)}{=} \sum_{l=0}^{\infty} n\sigma(n) v_{2l}(x) \int_0^\varepsilon L_\varepsilon(\widetilde{\eta})(r) r^{2l+n-1}\,dr
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} n\sigma(n) v_{2l}(x)\, \varepsilon^{2l-2} \int_0^1 L_1(\widetilde{\eta})(t) t^{2l+n-1}\, dt \\
&= \sum_{l=0}^{\infty} v_{2l}(x)\, \varepsilon^{2l-2} \cdot n\sigma(n) \int_{B(0,1)} L_1(\widetilde{\eta})(|y|)|y|^{2l}\, dy \\
&= \sum_{l=1}^{\infty} v_{2l}(x)\, \varepsilon^{2l-2} \int_{B(0,1)} \Delta\eta^1(y)\, y^{2l}\, dy
\end{aligned}
$$

since $\int_{B(0,1)} \Delta\eta^1(y)\, dy = \int_{S(0,1)} \frac{\partial\eta^1}{\partial n}(y)\, dS(y) = 0$.

So

$$\Delta\big(\eta^{\varepsilon} * u\big)(x) = \sum_{l=0}^{\infty} v_{2l+2}(x)\, \varepsilon^{2l} m_{2l+2}(\Delta\eta^1),$$

where $m_{2l}(\eta^1) = \int_{B(0,1)} \eta^1(y) y^{2l}\, dy$ for $l \in \mathbb{N}_0$.
Similarly for $k \in \mathbb{N}_0$ we get

$$\Delta^k\big(\eta^{\varepsilon} * u\big)(x) = \sum_{l=0}^{\infty} v_{2l+2k}(x) \cdot \varepsilon^{2l} m_{2l+2k}(\Delta^k \eta^1).$$

Note that $\Delta^k(\eta^\varepsilon * u)$ is distributionally convergent as $\varepsilon \to 0$. Hence

$$\begin{aligned} m_{2k}(\Delta^k \eta^1) v_{2k} &= \lim_{\varepsilon\to 0} \Delta^k(\eta^\varepsilon * u) \\ &= \Delta^k\big(\lim_{\varepsilon\to 0} \eta^\varepsilon * u\big) = \Delta^k v_0 \in D'(\Omega). \end{aligned}$$

Since $v_{2k} \in C(\Omega)$ applying the Weil lemma we conclude that $u = v_0 \in C^{2k}(\Omega)$.

Next for $0 \le l \le k$ we have
$\Delta^k v_0 = \Delta^{k-l}(\Delta^l v_0) = m_{2l}(\Delta^l \eta^1) \cdot \Delta^{k-l} v_{2l} \in C^0(\Omega)$.
So $v_{2l} \in C^{2k-2l}(\Omega)$.
Since k is arbitrary big we conclude that $v_{2l} \in C^\infty(\Omega)$ for $l \in \mathbb{N}_0$. $\square$

Theorem 6.4 *Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^0(\Omega)$. If there exist functions $u_{2l} \in C^0(\Omega)$ for $l \in \mathbb{N}_0$ and $\rho \in C^0(\Omega, \mathbb{R}_+)$ such that for any $x \in \Omega$ and $0 < R < \rho(x)$ it holds*

$$M(u;\, x, R) = \sum_{l=0}^{\infty} u_{2l}(x) R^{2l},$$

then u is real analytic on Ω and $4^l \left(\frac{n}{2}+1\right)_l l!\, u_{2l} = \Delta^l u$ for $l \in \mathbb{N}_0$.

Proof. Applying the relation between mean value functions M and N, we get for $x \in \Omega$ and $0 < R < \rho(x)$,

$$N(u;\, x, R) = \Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big)\Big(\sum\nolimits_{l=0}^{\infty} u_{2l}(x) R^{2l}\Big)$$
$$= \sum\nolimits_{l=0}^{\infty} \Big(\frac{2l}{n} + 1\Big) u_{2l}(x) R^{2l}.$$

Hence the assumptions of Lemma 6.3 are satisfied with $v_{2l} = \big(\frac{2l}{n} + 1\big) u_{2l}$ and so $u_{2l} \in C^{\infty}(\Omega)$ for $l \in \mathbb{N}_0$. Next we derive that $4^l \big(\frac{n}{2} + 1\big)_l l!\, u_{2l} = \Delta^l u$ and apply Theorem 6.2. □

7 Analytic functions on MMS

It is well known that the mean value characterization of harmonic functions can be used to define harmonic functions on metric measure spaces (MMS). Namely, let (X, ρ, μ) be a metric measure space with a metric ρ and a Borel regular measure μ which is positive on open sets and finite on bounded sets.

Then a continuous function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset X$ is said to be harmonic on Ω if for every $x \in \Omega$ and any closed ball $B(x, R) \subset \Omega$ it holds

$$u(x) = \frac{1}{\mu(B(x, R))} \int_{B(x,R)} u(y)\, d\mu(y).$$

If the measure is continuous with respect to the metric, then harmonic functions on MMS satisfy maximum principle, the Harnack type inequality and the Weierstrass and Montel convergence theorem, see [11].

Another approach to the theory of harmonic functions on MMS, based on variational methods, was proposed by Shanmugalingam [24].
Recently Alabern, Mateu and Verdera obtained in [2] a characterization of Sobolev spaces on $\mathbb{R}^n$ only in terms of the Euclidean metric and the Lebesgue measure which allowed them to define higher order Sobolev spaces on MMS.

We propose a definition of analytic functions on MMS.

Definition 1 Let (X, ρ, μ) be a metric measure space with a metric ρ and a Borel regular measure μ which is positive on open sets and finite on bounded sets. Let Ω be an open subset of X.

For any $x \in \Omega$ and $0 < R < \text{dist}(x, \partial\Omega)$ define a solid mean of a continuous function $u \in C^0(\Omega)$ by

$$M_X(u;\, x, R) = \frac{1}{\mu(B_\rho(x, R))} \int_{B_\rho(x,R)} u(y)\, d\mu(y).$$

Definition 2 Let (X, ρ, μ) be a metric measure space and Ω be an open subset of X.
Let $u : \Omega \to \mathbb{C}$ be a continuous function.
We say that u is (X, ρ, μ)-*analytic on* Ω and write $u \in \mathcal{A}_X(\Omega, \rho, \mu)$ if there exist functions $u_{2k} \in C^0(\Omega)$ for $k \in \mathbb{N}_0$ and $\epsilon \in C^0(\Omega, \mathbb{R}_+)$ such that for any $x \in \Omega$ and $0 < R < \epsilon(x)$ it holds

$$M_X(u; x, R) = \sum_{k=0}^{\infty} u_{2k}(x) R^{2k}.$$

The topology on $\mathcal{A}_X(\Omega, \rho, \mu)$ is given by

$$\mathcal{A}_X(\Omega, \rho, \mu) = \text{projlim}_{K \Subset \Omega} \text{indlim}_{\epsilon > 0} \mathcal{E}^{even}(K, \epsilon),$$

where

$$\begin{aligned} \mathcal{E}^{even}(K, \epsilon) = \{ & F \in C^0(K; C^\infty(-\epsilon, \epsilon)) : \\ & F(x, R) = F(x, -R) \ \text{ for } \ x \in K, |R| < \epsilon \ \text{ and} \\ & \|F\|_{K,\epsilon} = \sup_{k \in \mathbb{N}_0, x \in K} \frac{\left| \frac{\partial^k}{\partial R^k} F(x, 0) \right| \epsilon^k}{k!} < \infty \}. \end{aligned}$$

By the Theorem 6.4 the Pringsheim type theorem [14, Theorem] we get

Corollary 4 *Let* $X = \mathbb{R}^n$ *with the Euclidean metric* ρ *and the Lebesgue measure* λ*. Let* $\Omega \subset X$ *and* $u \in C^0(\Omega)$*. Then* u *is* (X, ρ, λ)*-analytic on* Ω *if and only if* u *is real analytic on* Ω*.*

Lemma 7.1 *Let u be a continuous function on an open subset Ω of a metric measure space (X, ρ, μ).*
If $M_X(u; x, R) = 0$ for any $x \in \Omega$ and $0 < R < \epsilon(x)$, then $u \equiv 0$.

Theorem 7.2 *Let u be (X, ρ, μ)-analytic on a connected open set $\Omega \subset X$.*
If u vanishes on an open set $U \subset \Omega$, then it vanishes on Ω.

Definition 3 The metric measure space (X, ρ, μ) is called *analytizable* if for any $x \in X$ there exist open sets $U \subset \mathbb{R}^n$, $\Omega \subset X$ and a homeomorphism $\Phi : U \to^{\text{onto}} \Omega$ such that for $y \in \Omega$ and R small enough

$$\Phi\big(B(\Phi^{-1}(y), R)\big) = B_\rho(y, R)$$

and for Borel sets $A \subset \Omega$

$$\mu(A) = |\Phi^{-1}(A)|.$$

Theorem 7.3 *Under the notations of Definition 3 let $u : \Omega \to \mathbb{C}$ be a continuous function.*
Then u is (X, ρ, μ)-analytic on Ω if and only if $u \circ \Phi$ is real analytic on $U = \Phi^{-1}(\Omega)$.

Hence if X is locally homeomorphic to $\mathbb{R}^n$, then the metrical properties of X-analytic functions can be derived from the analogous properties of real analytic functions.

Thank you for your attention!

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