Exponential growth order of sols. for Moser irreducible system and a counterexample for Barkatou's conjecture

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References

[Bar, ISSAC'10] M. Barkatou, Tutorial lecture at ISSAC '10, Münich.

[Kit, '83] K. Kitagawa, J. Math. Kyoto Univ., **23** (1983), 427 - 440.

[Miy, '12] M. Miyake, Funk. Ekvac., **55** (2012), 169 - 237

[M-I, '09] M. Miyake and K. Ichinobe, Funk. Ekvac., **52** (2009), 53 - 82.

[Mos, '60] J. Moser, Math. Z., **72** (1960), 1960 - 398.

[Vol, '60] R. Volevič, Dokl. Acad. Nauk SSSR, **132** (1960), 20 - 23
= Soviet Math. Dokl., **1** (1960), 458 - 461.

1 Introduction

Let us define a singular system $L = (p, A(z))$ of apparent Poincaré rank of $p \geq 1$ by

$$L \equiv (p, A(z)) := z^{p+1}\frac{d}{dz}I_N - A(z), \quad A(z) = \Big(a_{ij}(z)\Big) \in M_N(\mathbb{C}\{z\}). \tag{1.1}$$

Denote by $\rho(L) \in \mathbb{Q}_{\geq 0}$ the maximal exponential growth order in $|z|^{-1}$ of solutions $y(z)$ of homogeneous equation $Ly(z) = 0$, which we call the *irregularity* of L.

Purpose ; The purpose of this talk is to ask the possibility of characterizing $\rho(L)$ in terms of $A(z)$ for the Moser irreducible system $L = (p, A(z))$ which is *irregular singular* at $z = 0$.

By definition, the system L is *regular singular* (at $z = 0$) if every solution of $Ly = 0$ has at most polynomial growth in $|z|^{-1}$, and it is characterized by $\rho(L) = 0$. Then;

Theorem. *$L = (p, A(z))$ is regular singular $\iff$ it is reducible into a system of first kind at worst, the case when $O(A) \geq p$, by a reduction matrix in $GL_N(\mathbb{K}[z])$.*

This characterization is known as Horn's theorem [Hor, 1892], but an actual reduction was shown by J. Moser in [Mos, '60], where the notion of *irreducibility* was introduced.

Now we recall Moser's argument.

Let the Taylor expansion of $A(z)$ be

$$A(z) = \sum_{n=0}^{\infty} A_n z^{k+n}, \quad k = O(A) \geq 0 \ (\textit{order of zeros}), \ A_0 \neq O.$$

He defined two numbers $m(A)$ (*Moser's rank*) and $\mu(A)$ (*Moser's reduced rank*) by

$$m(A) = p - k + \frac{r}{N}, \quad r = \operatorname{rank} A_0 \geq 1,$$

$$\mu(A) = \min_{P(z) \in GL_N(\mathbb{K}[z])} \left\{ m(A_P) \ ; \ A_P(z) := P^{-1}AP - z^{p+1}P^{-1}P' \in M_N(\mathbb{C}\{z\}) \right\}.$$

- The case when $m(A) \leq 1$ is outside of our interest, because

$$m(A) \leq 1 \quad \Longleftrightarrow \quad k \geq p \quad \Longleftrightarrow \quad L \text{ is singular of first kind at worst.}$$

Definition (irreducibility) Let $m(A) > 1$. Then he defined;

$$L = (p, A(z)) \ \textit{is} \ \begin{cases} (\textit{Moser}) \ \textit{reducible} & \overset{\text{def.}}{\Longleftrightarrow} \quad m(A) > \mu(A), \\ (\textit{Moser}) \ \textit{irreducible} & \overset{\text{def.}}{\Longleftrightarrow} \quad m(A) = \mu(A). \end{cases}$$

He characterized the condition for the irreducibility in the following form.

Theorem [Mos]

$L = (p, A(z))$ is Moser irreducible $\iff$ $\mathcal{P}_A(\lambda) \not\equiv 0$, *where*

$$\mathcal{P}_A(\lambda) := \left[z^r \times \det \left(\lambda I_N - \frac{1}{z^{k+1}} A(z) \right) \right]_{z=0} = \left[z^r \times \det \left(\lambda I_N - (\frac{A_0}{z} + A_1) \right) \right]_{z=0},$$

which we call Moser's polynomial. In both inequalities, the conditions to hold the equalities are possible to characterize.

Theorem 1 (estimation of irregularity)

Let $L = (p, A(z))$ be Moser irreducible with nilpotent $A(0) = A_0$. Let us define

$$d = \deg_\lambda \mathcal{P}_A, \quad k_1 = \min\{k \,;\, A_0^k = O\}, \quad S_0 = \frac{N - d - r}{N - d}.$$

Then we have

$$p - S_0(L) \;\leq\; \rho(L) \;\leq\; p - \frac{1}{k_1}.$$

Remark. The first inequality is seen in [Bar, ISSAC'10, p.31] without proof.

The proof is done by applying the results in [Kit, '83] and in [M-I, '09] after asking the meaning of Moser irreducibility.

It is desirable to specify $\rho(L)$ exactly if possible, which is impossible in general situation. On this problem Barkatou stated a result under the condition ($p \geq (r+1) \times \frac{N-d-r}{N-d}$) on [Bar, p.33], and gave a conjecture that

Conjecture [Bar, ISSAC'10, p.35] For Moser irreducible system $L = (p, A(z))$ with nilpotent constant term A_0, it may hold that $\rho(L) = p - s_0(A)$ by

$$s_0(A) := \min_{1 \leq j \leq N} O(p_j)/j > 0, \qquad p_A(\lambda, z) := \det(\lambda I_N - A(z)) = \sum_{j=0}^{N} p_j(z)\lambda^{N-j}.$$

We give a counterexample to this conjecture. Precisely, for the given example, we specify the leading term of the Exp. factor $\Lambda(z)$ in the FFMS (formal fundamental matrix solution) $Y(z) = P(z)z^C e^{\Lambda(z)}$ of $Ly = 0$.

For this purpose we introduce a class of *surgery operations* for system transformations which consists of ① A_0-invariant transformation by matrices in $GL_N(\mathbb{C})$,

② $J(A_0)$ (Jordan type of A_0) change transformation by matrices in $GL_N(\mathbb{K}[z])$.

- By these surgery operations $\mathcal{P}_A(\lambda)$ is kept invariant, but the Newton polygon $\mathrm{N}(A)$ may change by $J(A_0)$-change transformation.

2 Preliminary considerations

2.1 On the Moser irreducibility

(1) Moser irreducibility and Newton polygon

Let $p_A(\lambda, z)$ denote the characteristic polynomial of $A(z)$ with $A_0 = A(0) \neq O$.

$$p_A(\lambda, z) = \lambda^N + \sum\nolimits_{k=1}^{N} p_j(z)\lambda^{N-j}, \quad p_j(z) = (-1)^j \sum_{1\le i_1<\cdots<i_j\le N} \det\Big(a_{i_k,i_\ell}(z)\Big)_{1\le k,\ell\le j}.$$

Let $Q(p_j) := \{(x,y) \in \mathbb{R}^2\ ;\ x \le j,\ y \ge O(p_j)\}$ $(p_0 \equiv 1)$. Then we define

$$\mathrm{N}(A) := \mathrm{Convex-hull}\left(\bigcup\nolimits_{j=0}^{N} Q(p_j)\right) \qquad (\mathit{Newton\ polygon\ of}\ A(z)).$$

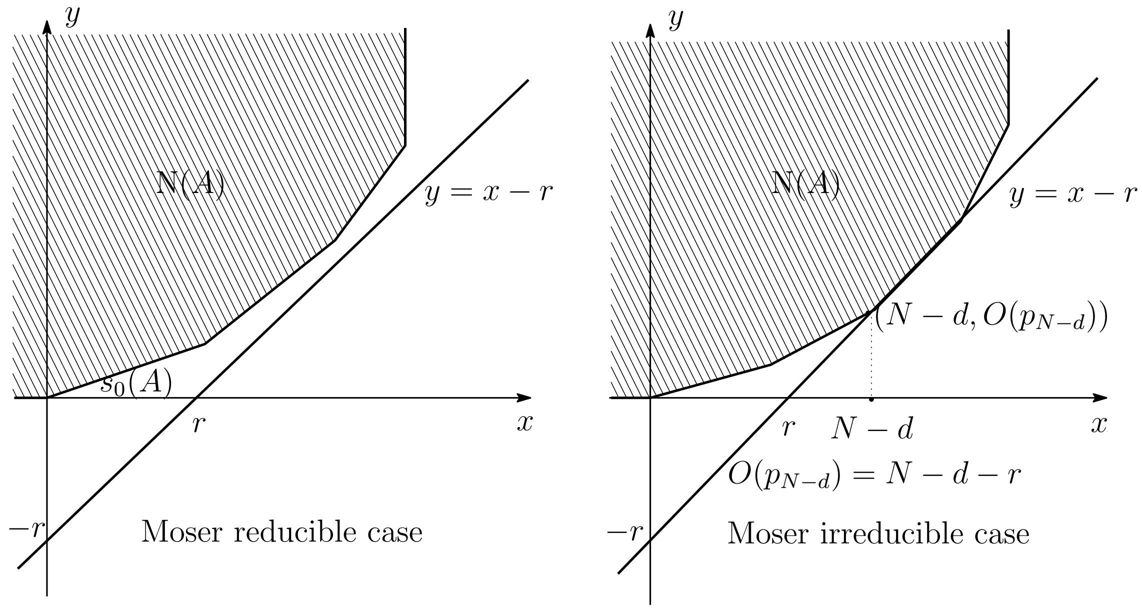
This shows that $s_0(A) = \min_{1\le j\le N} O(p_j)/j$ gives the smallest slope of sides of $\mathrm{N}(A)$.

Lemma 1 ($\mathcal{P}_A(\lambda)$ and $\mathrm{N}(A)$) *Let* $\mathrm{rank}A_0 = r$ (≥ 1). *Then ;*

(i) $\mathrm{N}(A)$ *lies in the region* $y \ge x - r$*.*

(ii) $\mathcal{P}_A(\lambda) \equiv 0$ *is equivalent that* $\mathrm{N}(A)$ *lies in the strictly upper region* $y > x - r$*.*

(iii) $\mathcal{P}_A(\lambda)$ *is determined from* $p_A(\lambda, z)$ *by the side or the vertex of* $\mathrm{N}(A)$ *on the line* $y = x - r$*.*



Proof. Put $A(z) = \sum_{n=0}^{\infty} A_n z^n$. The formula for $p_j(z)$ shows that

$$\operatorname{rank} A_0 = r \implies O(p_j) \geq \max\{j - r, 0\} \ (0 \leq j \leq N) \implies \text{(i)}.$$

The coefficient of z^{j-r} of $p_j(z)$ $(j \geq r)$ depends only on A_0 and A_1 denoted by $p_j(A_0, A_1)$. The expression $\det\bigl(\lambda I_N - z^{-1}A(z)\bigr) = z^{-N} p_A(z\lambda, z) = \sum_{j=0}^{N} z^{-j} p_j(z) \lambda^{N-j}$ shows

$$\begin{aligned} \mathcal{P}_A(\lambda) &:= \bigl[z^r \det(\lambda I_N - z^{-1}A(z))\bigr]_{z=0} \\ &= \sum_{j=0}^{N} \bigl[z^{r-j} p_j(z)\bigr]_{z=0} \lambda^{N-j} = \sum_{j=r}^{N} p_j(A_0, A_1) \lambda^{N-j}. \end{aligned} \tag{2.1}$$

□

Lemma 2 (invariance of $\mathcal{P}_A(\lambda)$) *$\mathcal{P}_A(\lambda)$ is invariant under the system transformations by matrices $P(z) \in GL_N(\mathbb{C}\{z\})$. In the Moser irreducible case, let $d = \deg_\lambda \mathcal{P}_A(\lambda)$. Then the vertex point $(N-d, N-d-r) \in \mathrm{N}(A)$ is the common vertex point of $\mathrm{N}(A_P)$ for every $P(z) \in GL_N(\mathbb{C}[z])$. This shows $s_0(A_P) \leq \frac{N-d-r}{N-d} =: S_0(L)$.*

Proof. It is trivial, since $A_P(z) = P^{-1}AP - z^{p+1}P^{-1}P' = P^{-1}AP + O(z^2)$ and $\operatorname{rank} A_P(0) = \operatorname{rank} A(0)$. $\square$

(2) Moser matrix $\mathcal{A}$ and Moser polynomial $\mathcal{P}_A(\lambda)$

We show a way for direct calculation of $\mathcal{P}_A(\lambda)$ by a sub-matrix of the coefficient matrix A_1 of z in the Taylor expansion $A(z) = \sum_{n=0}^{\infty} A_n z^n$. Let A_0 be a nilpotent matrix of Jordan canonical form. We define the *Jordan type* $J(A_0)$ by

$$A_0 = \bigoplus_{j=1}^{m_1} N_{k_j} \oplus O_{m_2}, \quad J(A_0) := (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2},$$

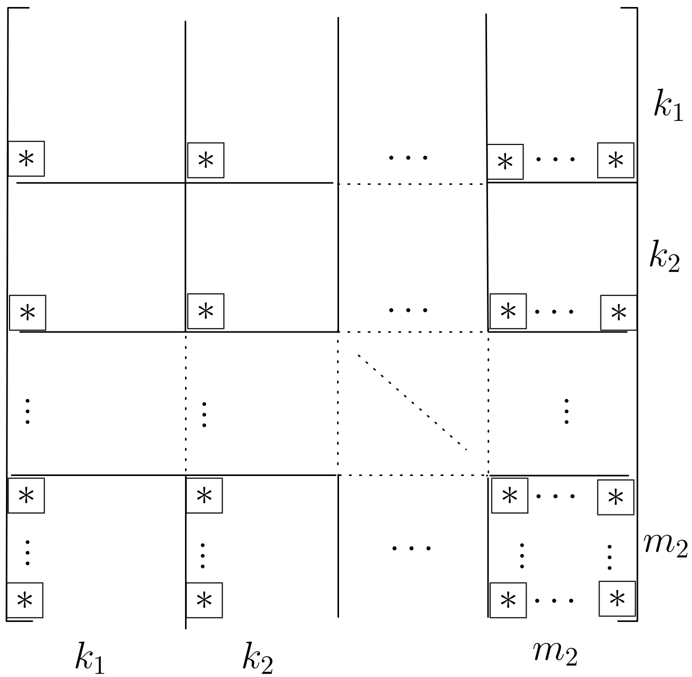
where $N_k \in M_k(\mathbb{C})$ $(k \geq 2)$ denotes the nilpotent Jordan cell of upper triangle form of rank $k-1$ and $O_m \in M_m(\mathbb{C})$ denotes the zero matrix. We define $\{k(j)\}_{j=0}^{m_1}$ by

$$k(0) := 0, \quad k(j) := \sum_{i=1}^{j} k_i \ \ (1 \leq j \leq m_1). \tag{2.2}$$

From $A_1 = \left(a_{ij}\right)$, the coefficient matrix of z, we choose $\{a^{[i,j]}\}_{1\le i,j\le m_1+m_2}$ by

$$a^{[i,j]} := \begin{cases} a_{k(i),k(j-1)+1}, & 1 \le i,\ j \le m_1, \\ a_{k(i),k(m_1)+t}, & 1 \le i \le m_1,\ j = m_1+t, \\ a_{k(m_1)+s,k(j-1)+1}, & i = m_1+s,\ 1 \le j \le m_1, \\ a_{k(m_1)+s,k(m_1)+t}, & i = m_1+s,\ j = m_1+t, \end{cases} \qquad (1 \le s,t \le m_2).$$

The elements $a^{[i,j]}$ are taken from the position $\boxed{*}$ in the following figure.

$A_1 =$ 

Now we define *Moser's matrix* $\mathcal{A} \in M_{m_1+m_2}(\mathbb{C})$ by

$$\mathcal{A} := \left(a^{[i,j]}\right) = \begin{bmatrix} \mathcal{A}^{[1,1]} & \mathcal{A}^{[1,2]} \\ \mathcal{A}^{[2,1]} & \mathcal{A}^{[2,2]} \end{bmatrix}, \quad \mathcal{A}^{[i,j]} \in M_{m_i \times m_j}(\mathbb{C}).$$

It is convenient to add the Jordan type $J(A_0)$ to $\mathcal{A}$ and we write it in the form,

$$\mathcal{A} = \begin{array}{c|cccccc} & k_1 & \cdots & k_{m_1} & 1 & \cdots & 1 \\ \hline k_1 & a^{[1,1]} & \cdots & a^{[1,m_1]} & a^{[1,m_1+1]} & \cdots & a^{[1,m_1+m_2]} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ k_{m_1} & a^{[m_1,1]} & \cdots & a^{[m_1,m_1]} & a^{[m_1,m_1+1]} & \cdots & a^{[m_1,m_1+m_2]} \\ 1 & a^{[m_1+1,1]} & \cdots & a^{[m_1+1,m_1]} & a^{[m_1+1,m_1+1]} & \cdots & a^{[m_1+1,m_1+m_2]} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a^{[m_1+m_2,1]} & \cdots & a^{[m_1+m_2,m_1]} & a^{[m_1+m_2,m_1+1]} & \cdots & a^{m_1+m_2,m_1+m_2]} \end{array}$$

Lemma 3 (calculation of $\mathcal{P}_A(\lambda)$) *The Moser polynomial is obtained by*

$$\mathcal{P}_A(\lambda) = \det\left[\{O_{m_1} \oplus \lambda I_{m_2}\} - \mathcal{A}\right] \ \left(= \sum_{j=0}^{m_2} q_j \lambda^{m_2-j}\right),$$

which shows that $\deg_\lambda \mathcal{P}_A(\lambda) \le m_2$. *The typical coefficients are* $q_0 = (-1)^{m_1} \det \mathcal{A}^{[1,1]}$, $q_{m_2} = (-1)^{m_1+m_2} \det \mathcal{A}$.

Proof. Let $J(A_0) = (k_1, \cdots, k_{m_1}, 1, \cdots, 1)$ and define

$$E_{A_0}(z) := \left(\bigoplus_{j=1}^{m_1} \{zI_{k_j-1} \oplus [1]\}\right) \oplus I_{m_2}, \quad \det E_{A_0}(z) = z^r.$$

For $B(\lambda, z) = \lambda I_N - z^{-1}A(z)$ we have $z^r \times \det B(\lambda, z) = \det [E_{A_0}(z) \circ B(\lambda, z)]$. Then letting $z = 0$ we get the formula after elementary operations for the determinant. $\square$

Remark. The similar result was given by Barkatou-Pflügel [2007, 2009] by different manner under the name L-matrix $L(A, \lambda)$.

2.2 On the irregularity $\rho(L)$ by [Kit] and [M-I]

For $A(z) = \left(a_{ij}(z)\right) \in M_N(\mathbb{C}\{z\})$, we put $r_{ij} = O(a_{ij}) \in \mathbb{N}_{\geq 0} \cup \{+\infty\}$. Then the Volevič weight $V(A) \in \mathbb{Q}_{\geq 0} \cup \{+\infty\}$ is defined by

$$V(A) := \min_{1\leq n\leq N} \min_{1\leq i_1<\cdots<i_n\leq N} \min_{\sigma\in S_n} \frac{1}{n}\sum_{k=1}^{n} r_{i_k, i_{\sigma(k)}} \ (\leq s_0(A)).$$

Let $V(A) < p$. Then we define;

- $L = (p, A(z))$ *is* non − degenerate in V − sense $\iff V(A) = s_0(A)$
- L *is* non − degenerate of full rank $\iff O(\det A(z)) = NV(A)$.

Lemma 4. *Let $V(A) < p$. Then we have;*

(i) $\rho(L) \leq p - V(A)$

(ii) $\rho(L) = p - V(A) \iff$ *L is non-degenerate in V-sense.*

(iii) *L is non-degenerate of full rank $\Longrightarrow$ the leading term of $\Lambda(z)$ in the FFMS is characterized from the leading term of the Puiseux expansion of the eigenvalues $\lambda(z)$; $p_A(\lambda, z) = 0$.*

Theorem 1 is an immediate consequence from the following

Theorem 2 [M-I, Th.A]

$L = (p, A(z))$ is of irregular singular type $\Longrightarrow$ $\exists P(z) \in GL_N(\mathbb{C}[z])$ s.t. $L_P :=$ $(p, A_P(z))$ is non-degenerate in V-sense of $V(A_P) < p$, $\rho(L) = p - V(A_P) > 0$.

The actual proof is done by using the following lemma by [Vol, '60].

Volevič's Lemma (cf. [Miy, '79]) *If $V(A) \in \mathbb{Q}_{\geq 0}$, then $\exists\ T = \{t_i\}_{j=1}^{N} \in \mathbb{Q}$ (V-numbers) s.t. $r_{ij} \geq t_i - t_j + V(A)$, $1 \leq i, j \leq N$.*

• $V(A) \geq p$ ($\Longrightarrow L = (p, A(z))$ is first kind in V-sense) $\Longrightarrow$ L is regular singular.

$\because$ $\exists T = \{t_i\} \in \mathbb{Z}$ s.t. $r_{ij} \geq t_i - t_j + p \Longrightarrow$ by $P(z) = z^T := \mathrm{diag}(z^{t_1}, \cdots, z^{t_N}) \in GL_N(\mathbb{K}[z])$, $L_P = (p, A_P(z))$ becomes a singular system of first kind at worst.

We write $a_{ij}(z) = \left\{ \overset{\circ}{a}_{ij} + o(1) \right\} z^{t_i - t_j + V(A)} \quad (\overset{\circ}{a}_{ij} = 0 \quad \text{if} \quad t_i - t_j + V(A) \notin \mathbb{N})$.

Then $\overset{\circ}{A} = \left(\overset{\circ}{a}_{ij} \right) \in M_N(\mathbb{C})$. is called the principal symbol of $A(z)$ in V-sense.

Lemma 5. (i) *$L = (p, A(z))$ is non-degenerate in V-sense $\iff$ $\overset{\circ}{A}$ is not a nilpotent matrix.*

(ii) *$L = (p, A(z))$ is non-degenerate of full rank* $\iff$ $\det \overset{\circ}{A} \neq 0$.

(iii) *$L = (p, A(z))$ is degenerate in V-sense of $V(A) < p$*

$\implies \exists P(z) \in GL_N(\mathbb{C}[z]) \quad s.t. \quad V(A_P) \geq V(A) + 1/N(N-1)$

$\implies \rho(L) \leq p - V(A) - 1/N(N-1)$.

In [Miy, '12], the author proved the irreducible decomposability of $L = (p, A(z))$ by non-degenerate subsystems of full rank for the irregular singular part, which tells us that the leading term of the exponential part $\Lambda(z)$ is possible to determine, but after making the irreducible decomposition.

In general framework, we can't foresee the leading term of the exponential factor $\Lambda(z)$ of the FFMS from the coefficient matrix $A(z)$ itself. It may possible to say that Barkatou's conjecture asks the possibility of this for Moser irreducible system $L = (p, A(z))$.

3 Proof of Theorem 1

Theorem 1. *For a Moser irreducible system* $L = (p, A(z))$*, assume that* $A_0 = A(0)$ *is a nilpotent matrix of Jordan type* $J(A_0) = (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}$ $(k_1 \geq \cdots \geq k_{m_1} \geq 2)$*. Let* $r = \mathrm{rank} A_0$ *and* $d = \deg_\lambda \mathcal{P}_A(\lambda)$*. Then we have*

$$p - S_0(L) \;\leq\; \rho(L) \;\leq\; p - \frac{1}{k_1}, \quad S_0(L) := \frac{N-d-r}{N-d}. \tag{3.1}$$

Let $\ell = \max\{j \; ; \; k_j = k_1\}$*. Then*

$$\rho(L) = p - 1/k_1 \quad \Longleftrightarrow \quad \mathcal{A}_\ell := \left(a^{[i,j]}\right)_{1 \leq i,j \leq \ell} \text{ is not nilpotent.} \tag{3.2}$$

Proof. The proof is immediate. Note that the point $(N-d, N-d-r)$ is a fixed vertex point of $\mathrm{N}(A_P)$ for $\forall P(z) \in GL_N(\mathbb{C}[z])$. Find $P(z) \in GL_N(\mathbb{C}[z])$ s.t. $L_P = (p, A_P(z))$ is non-degenerate in V-sense. Then this implies the first inequality;

$$s_0(A_P) \leq S_0(L) \quad \Longrightarrow \quad \rho(L) = \rho(L_P) = p - s_0(A_P) \geq p - S_0(L).$$

The second inequality is a consequence of Lemma 4 and that $V(A) \geq 1/k_1$.

For the proof of (3.2), it is sufficient to examine that

$$L = (p, A(z)) \text{ is non-degen. in V-sense with } V(A) = 1/k_1 \iff \mathcal{A}_\ell \text{ is not nilpotent}$$

Remark. (1) We take $P(z) \in GL_N(\mathbb{C}[z])$ s.t. $L_P = (p, A_P(z))$ is non-degen. in V-sense. Then $\rho(L) = p - S_0(L)$ if and only if $\mathrm{N}(A_P)$ has a unique side of slope $S_0(L)$ which is < 1. We only know this after reducing the system but not by given system.

(2) The Moser irreducibility is defined by the first two Taylor coefficients $\{A_0, A_1\}$ in the expansion $A(z) = \sum_{n=0}^{\infty} A_n z^n$, but the irregularity $\rho(L)$ depends on many Taylor coefficients even for the Moser irreducible system. If $\rho(L) = p - r$, we need Taylor coefficients of z^n with $n \leq N \times r$, that is, we can truncate the terms of z^n of $n > N \times r$.

Proposition. *Let $L = (p, A(z))$ be Moser irreducible. Then*

$$p + 1 > N \times S_0(L) \quad \Longrightarrow \quad \rho(L) = p - s_0(A). \tag{3.3}$$

Indeed, if we put $\rho(L) = p - r$ we know that $r \leq S_0(L)$. We take $P(z) \in GL_N(\mathbb{C}[z])$ s.t. $L_P = (p, A_P(z))$ is non-degen. in V-sense, i.e., $V(A_P) = r \leq S_0(L)$. Since $A_P = P^{-1}AP - z^{p+1}P^{-1}P'$, the part $z^{p+1}P^{-1}P'$ is negligible for the determination of $s_0(A_P)$ and we have $s_0(A_P) = s_0(A)$. Actually, this is proved from that we can find the V-numbers $T = \{t_i\}$ such that $\sigma(T) := \max |t_i - t_j| \leq (N-1) \times V(A)$ (cf. [M-I, '09]. □

4 A counterexample for Barkatou's conjecture

4.1 Introduction of system

We give a Moser irreducible system $L = (p, A(z))$ $(p \geq 1)$ of $A(z) \in M_9(\mathbb{C}[z])$ by

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \boxed{0} & z & 0 & z & 0 & \boxed{z} & 0 & \boxed{0} & \boxed{0} \\ 0 & z^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{0} & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{z} & \boxed{0} \\ \boxed{0} & 0 & 0 & 0 & 0 & \boxed{0} & -z & \boxed{0} & \boxed{z} \\ \boxed{z} & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{0} & \boxed{0} \end{pmatrix}, \quad V(A) = \frac{1}{4} < \frac{4}{9} = s_0(A), \quad \operatorname{rank} A_0 = 5,$$

which means that the system is degenerate in V-sense. In fact, the characteristic polynomial is given by

$$p_A(\lambda, z) = \lambda^9 - z\lambda^8 + (-2z + z^2)\lambda^7 - z^3\lambda^6 + (z^2 - 2z^3)\lambda^5 + z^4\lambda^3 - z^4,$$

and $\mathrm{N}(A)$ has only one side $\overline{(0,0),(9,4)}$ of slope $4/9$.

The Moser matrix and Moser polynomials are given by

$$\mathcal{A} = \begin{array}{c|cccc} & 5 & 2 & 1 & 1 \\ \hline 5 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array}, \quad \mathcal{P}_A(\lambda) = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = -1, \quad d = \deg_\lambda \mathcal{P}_A = 0.$$

Note that $S_0(L) = \frac{N-d-r}{N-d} = \frac{4}{9} = s_0(A) > V(A) = \frac{1}{4}$ which implies $\rho(L) < 1 - 1/4$.

We shall prove ;

(4.1)
$$\rho(L) = 1 - \frac{2}{5} > 1 - s_0(A) = 1 - \frac{4}{9}.$$

Let $\overset{\circ}{\Lambda}(z)$ be the leading term of the Exp. factor $\Lambda(z)$ of the FFMS. Then

(4.2) $\overset{\circ}{\Lambda}(z) = \mathrm{diag}\left(\cdots, \frac{\alpha_j}{-3/5} z^{-3/5}, \cdots, \frac{\beta_k}{-1/2} z^{-1/2}, \cdots\right), \ 1 \le j \le 5, \ 1 \le k \le 4,$

by $\alpha_j^5 + 1 = 0$ and $\beta_k^4 - 1 = 0$.

4.2 Reduction into a non-degenerate system in V-sense

We make a system transformation by the following matrix.

$$P_1(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{z} & 0 & \boxed{z} & 0 & \boxed{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = D_9(5; z) \circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $D_9(5; z) := \mathrm{diag}(1, 1, 1, 1, z, 1, 1, 1, 1) \in GL_9(\mathbb{K}[z])$.

After the system transformation, the Jordan type $J(A_0) = (5, 2, 1, 1)$ is changed in the reduced system $L_1 = (1, A_1(z))$, but $\mathrm{rank}\, A_0$ is kept invariant.

We call such transformation a $J(A_0)$-change transformation.

Actually $A_1(z)$ becomes

$$
A_1(z) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\boxed{0} & 0 & z & 0 & \boxed{z} & 0 & 0 & \boxed{0} & \boxed{0} \\
-z-z^2 & -z & -z & 0 & -z & 1 & 0 & 0 & 0 \\
0 & z^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\boxed{0} & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{z} & \boxed{0} \\
\boxed{0} & 0 & 0 & 0 & \boxed{0} & 0 & -z & \boxed{0} & \boxed{z} \\
\boxed{z} & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{0} & \boxed{0}
\end{pmatrix}, \qquad V(A_1) = \frac{2}{5} \tag{4.3}
$$

$$
\mathcal{A}_1 = \begin{array}{c|cccc}
 & 4 & 3 & 1 & 1 \\
\hline
4 & 0 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0
\end{array}, \qquad \mathcal{P}_{A_1}(\lambda) = -1 \quad \text{(Moser polynomial is invariant).}
$$

The characteristic polynomial $p_{A_1}(\lambda, z) = \sum_{j=0}^{9} p_j(z)\lambda^{9-j}$ is obtained by

$$
\{p_j(z)\}_{j=0}^{9} = \left\{\underline{1}, 0, -2z, -z^2, -z^4 - 3z^3 + z^2, \underline{z^2} - z^4, z^4 - z^5, z^4, 0, \underline{-z^4}\right\},
$$

$$
\{(j, O(p_j)\}_{j=0}^{9} = \left\{\underline{(0,0)}, (1,\infty), (2,1), (3,2), (4,2), \underline{(5,2)}, (6,4), (7,4), (8,\infty), \underline{(9,4)}\right\}.
$$

Hence $\mathrm{N}(A_1)$ has two sides

$$\overline{(0,0),(5,2)} \text{ of slope } \frac{2}{5} = s_0(A_1) < s_0(A), \qquad \overline{(5,2),(9,4)} \text{ of slope } \frac{1}{2}.$$

This implies (4.1), since $L_1 = (1, A_1(z))$ is non-degenerate in V-sense of $V(A_1) = 2/5$,

$$\rho(L) = \rho(L_1) = 1 - \frac{2}{5} > 1 - s_0(A) = 1 - \frac{4}{9}. \tag{4.1}$$

In order to prove (4.2) on the leading term of the Exp. factor $\Lambda(z)$ of the FFMS, we need the Puiseux expansion of the eigenvalues of $A_1(z)$, i.e., $p_{A_1}(\lambda, z) = 0$.

By the property of $\mathrm{N}(A_1)$, the leading term of the Puiseux expansion is obtained from $\lambda^9 + z^2\lambda^4 - z^4 = 0$. Or equivalently they are obtained from

$$\lambda^5 + z^2 = 0 \quad \text{and} \quad \lambda^4 - z^2 = 0.$$

This is the reason why the form in (4.2) appear. We shall prove it in concrete form by reducing $L_1 = (1, A_1(z))$ into a decomposable system by surgery operations for the system transformations.

Remark (system reduction in V-style) The system reduction in V-style is made by using the null vectors for the principal matrix in V-sense. In this case, we take

$P(z) = E_{(5,1)}(z) \in GL_9(\mathbb{C}[z])$ which differs from I_9 that z is located on $(5,1)$-position,

$$P(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the reduced system $L_P = (1, A_P(z))$ becomes

$$A_P(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -z^2 & 0 & 0 & z & 0 & z & 0 & 0 & 0 \\ 0 & z^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & z \\ z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_{\mathcal{P}} = \begin{array}{c|cccc} & 5 & 2 & 1 & 1 \\ \hline 5 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array}, \quad V(A_P) = \frac{2}{5},$$

$$\mathcal{A}_{\mathcal{P}} = \begin{array}{c|cccc} & 5 & 2 & 1 & 1 \\ \hline 5 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array} .$$

Let the characteristic polynomial of $A_P(z)$ be $p_{A_P}(\lambda, z) = \sum_{j=0}^{9} p_j(z)\lambda^{9-j}$. Then ;

$$\{p_j(z)\}_{j=0}^{9} = \left\{\underline{1}, -z, z^2 - 2z, -z^3, z^2 - 2z^3, \underline{z^2}, z^4, z^4, 0, \underline{-z^4}\right\},$$

$$\{(j, O(p_j))\}_{j=0}^{9} = \{\underline{(0,0)}, (1,1), (2,1), (3,3), (4,2), \underline{(5,2)}, (6,4), (7,4), (8,\infty), \underline{(9,4)}\}.$$

Hence $V(A_P) = s_0(A_P) = 2/5$ (non-degenerate in V-sense), and $\mathrm{N}(A_P) = \mathrm{N}(A_1)$.

4.3 Reduction of $L_1 = (1, A_1)$ to get the Exp. factor $\Lambda(z)$

(1) Idea of the system transformation

In order to obtain the leading term $\overset{\circ}{\Lambda}(z)$ of the FFMS given by (4.2), we need to reduce the system $L_1 = (1, A_1(z))$ into a system $M = (1, B(z))$ which satisfies the following property;

- Moser matrix $\mathcal{B}$ of $B(z)$ has a decomposition $\mathcal{B}^{(1)} \oplus \mathcal{B}^{(2)}$, with the following property;

Let $B(z) = \left(B^{(i,j)}(z)\right)_{i,j=1,2}$ be the associated block decomposition to that of $\mathcal{B}$, and define $M^{(i)} = (1, B^{(i,i)}(z))$ $(i = 1, 2)$. Then each $M^{(i)}$ non-degenerate of full rank of $V(B^{(1,1)}) = 2/5$ and $V(B^{(2,2)}) = 1/2$, respectively. Furthermore, we ask that the leading terms of of the eigenvalues of $B^{(i,i)}(z)$ are obtained by $\lambda^5 + z^2 = 0$and , $\lambda^4 - z^2 = 0$.

This means that for the system reduction, we ask the invariance of Moser polynomial and rank A_0. (In general we do not ask the invariance of the sides of Newton polygon.)

We call such system transformations *surgery operations*. In our situation, the operations consist of

① A_0-invariant transformations by matrices in $GL_N(\mathbb{C})$

② $J(A_0)$-change transformation ($J(A_0)$ is changed but rank A_0 is invariant) is made by matrices in $GL_N(\mathbb{K}[z])$.

We start with the system $L_1 = (1, A_1(z))$ which is Moser irreducible and non-degenerate in V-sense of $V(A_1) = 2/5$, but not of full rank.

$$A_1(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & 0 & z & 0 & \boxed{z} & 0 & 0 & \boxed{0} & \boxed{0} \\ -z-z^2 & -z & -z & 0 & -z & 1 & 0 & 0 & 0 \\ 0 & z^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{0} & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{z} & \boxed{0} \\ \boxed{0} & 0 & 0 & 0 & \boxed{0} & 0 & -z & \boxed{0} & \boxed{z} \\ \boxed{z} & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}, \quad \mathcal{A}_1 = \begin{array}{c|cccc} & 4 & 3 & 1 & 1 \\ \hline 4 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array}$$

- Keep in mind that we want to get subsystems of non-degenerate of full rank !

This suggests to kill 1 on $(5, 6)$-position by $J(A_0)$-change transformation.

(2) $J(A_0)$-change transformation

We make a $J(A_0)$-change transformation for $L_1 = (1, A_1(z))$ by

$$P_2(z) = D_9(7; z) \circ D_9(6; z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the reduced system $L_2 = (1, A_2(z))$ becomes

$$A_2(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & 0 & z & 0 & \boxed{z} & \boxed{0} & 0 & 0 & \boxed{0} \\ \boxed{-z-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z} & 0 & 0 & \boxed{0} \\ 0 & z & 0 & 0 & 0 & -z & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1 & 0 \\ \boxed{0} & 0 & 0 & 0 & \boxed{0} & \boxed{0} & -z^2 & 0 & \boxed{z} \\ \boxed{z} & 0 & 0 & 0 & \boxed{0} & \boxed{0} & 0 & 0 & \boxed{0} \end{pmatrix}, \quad \mathcal{A}_2 = \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & 0 & 1 & 0 & 0 \\ 1 & \underline{-1} & -1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 1 & \boxed{1} & 0 & 0 & 0 \end{array}.$$

(3) A_0-invariant transformation

Our reduction is to make the decomposition of $\mathcal{A}_2$.
We kill 1 on $(4,1)$-position of $\mathcal{A}_2$ by the second row by the following A_0-invariant matrix,

$$P_3 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&\boxed{-1}&0&0&0&1 \end{pmatrix} \underset{\text{put}}{=} \begin{array}{c|cccc} & 4 & 1 & 3 & 1\\ \hline 4 & I_{4\times4} & 0 & 0 & 0\\ 1 & 0 & I_{1\times1} & 0 & 0\\ 3 & 0 & 0 & I_{3\times3} & 0\\ 1 & 0 & \boxed{-I_{1\times1}} & 0 & I_{1\times1} \end{array}.$$

Then the reduced system $L_3 = (1, A_3(z))$ becomes

$$A_3(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ \boxed{0} & 0 & z & 0 & \boxed{z} & \boxed{0} & 0 & 0 & \boxed{0}\\ \boxed{-z-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z} & 0 & 0 & \boxed{0}\\ 0 & z & 0 & 0 & 0 & -z & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1 & 0\\ \boxed{0} & 0 & 0 & 0 & \boxed{-z} & \boxed{0} & -z^2 & 0 & \boxed{z}\\ \boxed{-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z} & 0 & 0 & \boxed{0} \end{pmatrix}, \quad \mathcal{A}_3 = \begin{array}{c|cccc} & 4 & 1 & 3 & 1\\ \hline 4 & 0 & 1 & 0 & 0\\ 1 & \underline{-1} & -1 & \boxed{1} & 0\\ 3 & 0 & -1 & 0 & 1\\ 1 & 0 & -1 & 1 & 0 \end{array}.$$

(4) A_0-invariant transformation

We kill 1 on the (2,3)-position of $\mathcal{A}_3$ by the first column by

$$P_4 = \begin{pmatrix} 1&0&0&0&0&\boxed{1}&0&0&0\\ 0&1&0&0&0&0&\boxed{1}&0&0\\ 0&0&1&0&0&0&0&\boxed{1}&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \underset{put}{=} \begin{array}{c|cccc} & 4 & 1 & 3 & 1\\ \hline 4 & I_{4\times4} & 0 & I_{4\times3} & 0\\ 1 & 0 & I_{1\times1} & 0 & 0\\ 3 & 0 & 0 & I_{3\times3} & 0\\ 1 & 0 & 0 & 0 & I_{1\times1} \end{array}.$$

Then the reduced system $L_4 = (1, A_4(z))$ becomes

$$A_4(z) = \begin{pmatrix} 0 & 1-z & 0 & 0 & 0 & z & -z & 0 & 0\\ 0 & z & 1 & 0 & 0 & 0 & 2z & 0 & 0\\ z^2 & z & 0 & 1 & z & z^2 & z+z^2 & 0 & -z\\ \boxed{0} & 0 & z & 0 & \boxed{z} & \boxed{0} & 0 & z & \boxed{0}\\ \boxed{-z-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{-z^2} & -z & -z & \boxed{0}\\ 0 & z & 0 & 0 & 0 & -z & 1+z & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1 & 0\\ \boxed{0} & 0 & 0 & 0 & \boxed{-z} & \boxed{0} & -z^2 & 0 & \boxed{z}\\ \boxed{-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z-z^2} & -z & -z & \boxed{0} \end{pmatrix}, \quad \mathcal{A}_4 = \begin{array}{c|cccc} & 4 & 1 & 3 & 1\\ \hline 4 & 0 & \underline{1} & 0 & 0\\ 1 & -1 & \boxed{-1} & 0 & 0\\ 3 & 0 & \boxed{-1} & 0 & 1\\ 1 & 0 & \boxed{-1} & 1 & 0 \end{array}.$$

(5) A_0-invariant transformation (completion of decomposition)

We kill the -1's on the second column of $\mathcal{A}_4$ by the first row by

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{-1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{\text{put}}{=} \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & I_{4\times 4} & 0 & 0 & 0 \\ 1 & -I_{1\times 4} & I_{1\times 1} & 0 & 0 \\ 3 & -I_{3\times 4} & 0 & I_{3\times 3} & 0 \\ 1 & -I_{1\times 4} & 0 & 0 & I_{1\times 1} \end{array}.$$

Then the reduced system $L_5 = (1, A_5(z))$ becomes

$$A_5(z) = \begin{pmatrix} 0 & 1-2z & z & 0 & 0 & z & -z & 0 & 0 \\ 0 & z & 1-2z & 0 & 0 & 0 & 2z & 0 & 0 \\ z^2 & z-z^2 & -z-z^2 & 1 & z & z^2 & z+z^2 & 0 & -z \\ \boxed{0} & 0 & z & -2z & \boxed{z} & \boxed{0} & 0 & z & \boxed{0} \\ \boxed{-z-z^2} & z^2-z & z & 0 & \boxed{0} & \boxed{-z^2} & -z & 0 & \boxed{0} \\ 0 & 3z & -3z & 0 & 0 & -z & 1+3z & 0 & 0 \\ z^2 & z-z^2 & -z^2 & 0 & z & z^2 & z^2 & 1 & -z \\ \boxed{0} & 0 & z+z^2 & -2z & \boxed{0} & \boxed{0} & -z^2 & z & \boxed{z} \\ \boxed{-z^2} & z^2-2z & z & 0 & \boxed{0} & \boxed{z-z^2} & -z & 0 & \boxed{0} \end{pmatrix},$$

and the Moser matrix becomes a cyclic decomposition,

$$\mathcal{A}_5 = \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array} = \begin{array}{c|cc} & 4 & 1 \\ \hline 4 & 0 & 1 \\ 1 & -1 & 0 \end{array} \oplus \begin{array}{c|cc} & 3 & 1 \\ \hline 3 & 0 & 1 \\ 1 & 1 & 0 \end{array}.$$

Associated to the above decomposition the corresponding coefficient matrices of the subsystems $L_{5,j} = (1, A_{5,j}(z))$ $(j = 1, 2)$ are given by

$$A_{5,1}(z) = \begin{pmatrix} 0 & 1-2z & z & 0 & 0 \\ 0 & z & 1-2z & 0 & 0 \\ z^2 & z-z^2 & -z-z^2 & 1 & z \\ \boxed{0} & 0 & z & -2z & \boxed{z} \\ \boxed{-z-z^2} & z^2-z & z & 0 & \boxed{0} \end{pmatrix}, \qquad V(A_{5,1}) = s_0(A_{5,1}) = \frac{2}{5}$$

$$A_{5,2}(z) = \begin{pmatrix} -z & 1+3z & 0 & 0 \\ z^2 & z^2 & 1 & -z \\ \boxed{0} & -z^2 & z & \boxed{z} \\ \boxed{z-z^2} & -z & 0 & \boxed{0} \end{pmatrix}, \qquad V(A_{5,2}) = s_0(A_{5,2}) = \frac{1}{2}.$$

Then it is easily examined that the systems $L_{5,j}$ are non-degenerate of full rank. The principal matrices in V-sense of $A_{5,j}(z)$ are obtained by

$$\overset{\circ}{A}_{5,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overset{\circ}{A}_{5,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

From these observations we know that the leading term $\overset{\circ}{\Lambda}(z)$ of the FFSM is obtained as given in (4.2).

If one wants to see this fact, we make flat the systemL_5 by introducing the fractional power of z from the V-numbers for the subsystems $L_{5,j}$, and we apply the usual reduction procedure in the book by Wasow [Was].

4.4 Flatten of system by the V-numbers of $L_{5,i}$ $(i = 1, 2)$

The V-numbers $T^{(1)} = \{t_j^{(1)}\}_{j=1}^5$ for the matrix $A_{5,1}(z)$ is given by

$$t_1^{(1)} = 0,\ t_2^{(1)} = \frac{2}{5},\ t_3^{(1)} = \frac{4}{5},\ t_4^{(1)} = \frac{6}{5},\ t_5^{(1)} = \frac{3}{5}.$$

Similarly, for $A_{5,2}(z)$ we have the V-numbers $T^{(2)} = \{t_j^{(2)}\}_{j=1}^4$ by

$$t_1^{(2)} = 0,\ t_2^{(2)} = \frac{1}{2},\ t_3^{(2)} = 1,\ t_4^{(2)} = \frac{1}{2}.$$

We make flatten the system L_5 by taking the system transformation by

$$P_6 = z^{T^{(1)}} \oplus z^{T^{(2)}} = \mathrm{diag}\left(1, z^{2/5}, z^{4/5}, z^{6/5}, z^{3/5}, 1, z^{1/2}, z, z^{1/2}\right).$$

Then the reduced system $L_6 = (1, A_6(z))$ becomes that $A_6(z) =$

$$\begin{pmatrix} 0 & z^{2/5}-2z^{7/5} & z^{9/5} & 0 & 0 & z & -z^{3/2} & 0 & 0 \\ 0 & \frac{3z}{5} & z^{2/5}-2z^{7/5} & 0 & 0 & 0 & 2z^{11/10} & 0 & 0 \\ z^{6/5} & z^{3/5}-z^{8/5} & -z^2-\frac{9z}{5} & z^{2/5} & z^{4/5} & z^{6/5} & z^{17/10}+z^{7/10} & 0 & -z^{7/10} \\ 0 & 0 & z^{3/5} & -\frac{16z}{5} & z^{2/5} & 0 & 0 & z^{4/5} & 0 \\ -z^{7/5}-z^{2/5} & z^{9/5}-z^{4/5} & z^{6/5} & 0 & -\frac{3z}{5} & -z^{7/5} & -z^{9/10} & 0 & 0 \\ 0 & 3z^{7/5} & -3z^{9/5} & 0 & 0 & -z & 3z^{3/2}+\sqrt{z} & 0 & 0 \\ z^{3/2} & z^{9/10}-z^{19/10} & -z^{23/10} & 0 & z^{11/10} & z^{3/2} & z^2-\frac{z}{2} & \sqrt{z} & -z \\ 0 & 0 & z^{9/5}+z^{4/5} & -2z^{6/5} & 0 & 0 & -z^{3/2} & 0 & \sqrt{z} \\ -z^{3/2} & z^{19/10}-2z^{9/10} & z^{13/10} & 0 & 0 & \sqrt{z}-z^{3/2} & -z & 0 & -\frac{z}{2} \end{pmatrix}$$

The matrix of order of zeros in the fractional power of entries is

$$\frac{1}{10}\begin{pmatrix} \infty & \boxed{4} & 18 & \infty & \infty & 10 & 15 & \infty & \infty \\ \infty & 10 & \boxed{4} & \infty & \infty & \infty & 11 & \infty & \infty \\ 12 & 6 & 10 & \boxed{4} & 8 & 12 & 7 & \infty & 7 \\ \infty & \infty & 6 & 10 & \boxed{4} & \infty & \infty & 8 & \infty \\ \boxed{4} & 8 & 12 & \infty & 10 & 14 & 9 & \infty & \infty \\ \infty & 14 & 18 & \infty & \infty & 10 & \boxed{5} & \infty & \infty \\ 15 & 9 & 23 & \infty & 11 & 15 & 10 & \boxed{5} & 10 \\ \infty & \infty & 8 & 12 & \infty & \infty & 15 & \infty & \boxed{5} \\ 15 & 9 & 13 & \infty & \infty & \boxed{5} & 10 & \infty & 10 \end{pmatrix}.$$

References

[Bar] M. Barkatou, Symbolic Methods for Solving Systems of Linear Ordinary Differential Equations (II), 40 pp, Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC Germany 2010.

[B-P1] M. Barkatou and E. Pflügel, Computing super-irreducible forms of systems of linear differential equations via Moser-reduction: A new approach, Proceedings of ISSAC'07, ACM Press, Waterloo, Canada, pp. 1-8.

[B-P2] ———, On the Moser- and super-reduction algorithms of systems of linear differential equations and their complexity, J. Symbolic Computation, **44** (2009), 1017 - 1036.

[Kit] K. Kitagawa, L'irrégularité en un point singulier d'un systèmes d'équations différentielles linéaires d'ordre 1, J. Math. Kyoto Univ., **23** (1983), 427 - 440.

[Miy] M. Miyake, Newton polygon and Gevrey hierarchy in the index formula for a singular system of ordinary differential equations, Funk. Ekvac., **55** (2009), 169 - 237.

[M-I] M. Miyake and K. Ichinobe, Irregularity for singular system of ordinary differential equations in complex domain, Funk. Ekvac., **52** (2009), 53 - 82.

[Mos] J. Moser, The order of singularity in Fuch's theory, Math. Z., **72** (1960), 1960 - 398.

[Vol] R. Volevič, On general system of ordinary differential equations, Dokl. Acad. Nauk SSSR, **132** (1960), 20 - 23 = Soviet Math. Dokl., **1** (1960), 458 - 461.

[Was] W. Wasow, *Asymptotic Expansion of of Ordinary Differential Equations*, John Wiley, New York, 1965.