On coupling equations and their reversibility

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Tahara's coupling theory

The coupling theory is a theory of a class of transformations between some nonlinear PDE's in complex domains, introduced by Tahara.

The idea is to give a transformation $u(t,x) \mapsto w(t,x) = \Phi[u](t,x)$ between solutions to two PDE's, of form:

$$\Phi : u \mapsto w, \quad w(t,x) = \phi(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x),\frac{\partial^2 u}{\partial x^2}(t,x),\dots),$$

given in terms of "holomorphic functions of infinitely many variables" $\phi(t,x,z) = \phi(t,x,z_0,z_1,\dots)$, $(z = (z_i)_{i\in\mathbb{N}})$.

Tahara '07: for equations of normal form in the t variable.
Tahara '09, '13: for equations of Briot-Bouquet type.

coupling for equations of normal form I

Consider two equations of normal form

$$\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}) \quad (\mathrm{F}), \qquad \frac{\partial w}{\partial t} = G(t, x, w, \frac{\partial w}{\partial x}) \quad (\mathrm{G}),$$

with $F(t, x, z_0, z_1)$, $G(t, x, z_0, z_1) \in \mathcal{O}_{\mathbb{C}^4,0}$. Then, for a correspondence

$$\Phi : u \mapsto w, \quad w(t, x) = \phi(t, x, u(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial^2 u}{\partial x^2}(t, x), \dots),$$

to transform solutions to (F) into those to (G), $\phi(t, x, z)$ should formally satisfy the *coupling equation*:

$$\frac{\partial \phi}{\partial t} + \sum_{m \geq 0} D^m[F](t, x, z_0, \dots, z_{m+1}) \frac{\partial \phi}{\partial z_m} = G(t, x, \phi, D[\phi]). \quad (\mathrm{F} \to \mathrm{G})$$

coupling for equations of normal form II

Here D is a formal vector field defined by

$$D := \frac{\partial}{\partial x} + \sum_{m\geq 0} z_{m+1}\frac{\partial}{\partial z_m},$$

which is expected to satisfy the *chain rule*:

$$\frac{\partial}{\partial x}\{\phi(t,x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\cdots)\} = D[\phi](t,x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\cdots).$$

We want to solve the coupling equation (F $\to$ G) under the additional initial value condition $\phi(0,x,z) = z_0$. ($\to \Phi[u](0,x) \equiv u(0,x)$.)

coupling for equations of normal form III

In Tahara '07, he treated ϕ as a formal power series of form

$$\phi = z_0 + \sum_{k\geq 1} \phi_k(x, z_0, \dots, z_k)t^k \in \sum_{k\geq 0} \mathcal{O}_{|x|\leq R}[[z_0, \dots, z_k]]t^k,$$

and discussed in the case $G \equiv 0$ (i.e., coupling $\frac{\partial u}{\partial t} = F \leftrightarrow \frac{\partial w}{\partial t} = 0$):

- the unique existence of a formal power series solution ϕ to

$$\partial_t \phi + \sum_{m\geq 0} D^m[F] \cdot \partial_{z_m} \phi = 0. \qquad (\mathrm{F} \to 0)$$

- the estimate of ϕ so that $w = \phi(t, x, u, \partial_x u, \dots)$ makes sense.
- similar statements for the reversed equation:

$$\partial_t \psi = F(t, x, \psi, D[\psi]). \qquad (0 \to \mathrm{F})$$

- "reversibility" of ϕ and ψ, (i.e., Φ and Ψ are inverses each other).

motivation and overview I

The original coupling theory seemed to me to have some redundancy, coming from

- dependence on formal power series expansions,
- not symmetric discussions.

We report our recent study on couplings for equations of normal form, with

some merits

1. a new framework based on an infinite dimensional holomorphy
   $\rightarrow$ a robust action of D and chain rules
2. a unique solvability result by a functional analytic method
   $\rightarrow$ free from the requirement of holomorphic dependence in t
3. unified discussions and the notion of coupling maps
   $\rightarrow$ the native reversibility (independent of the class of operands)

overview II

t may be a (complex / complex sectorial / real / real positive) variable. Today we mainly work on the real positive case, i.e., a coupling between

$$\partial_t u = F(t,x,u,\partial_x u) \quad (\mathrm{F}), \qquad \partial_t w = G(t,x,w,\partial_x w) \quad (\mathrm{G}),$$

where F and G are continuous in $t \in [0, r_0]$ and holomorphic in (x, z_0, z_1) on a polydisc in $\mathbb{C}^3_{(x,z_0,z_1)}$.

an application

The unique solutions ϕ and ψ to coupling equations (F $\to$ G) and (G $\to$ F) define local transformations Φ and Ψ along $\mathbb{C}_x \simeq \{(t,x) \mid t = 0\}$ between solution spaces to (F) and (G) with "small" initial data at $t = 0$, which are inverses each other.

At this moment, the "smallness" for the initial data is required not only on $|\partial_x^i u(0,x)|$ $(i = 0, 1)$, but also on $|\partial_x^2 u(0,x)|$. We hope to resolve this technical difficulty soon.

sequence spaces and topologies

We regard $\phi(t,x,z) = \phi(t,x,z_0,z_1,\dots)$ as a function defined on a subset of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^{\mathbb{N}}$ or $\mathbb{R}_t \times \mathbb{C}_x \times \mathbb{C}_z^{\mathbb{N}}$. For this purpose, we take some locally convex spaces X continuously embedded as

$$\mathbb{C}_z^{(\mathbb{N})} \hookrightarrow X \hookrightarrow \mathbb{C}_z^{\mathbb{N}},$$

$$\mathbb{C}^{\mathbb{N}} := \{z = (z_i)_{i\in\mathbb{N}} \mid z_i \in \mathbb{C}\}, \quad \mathbb{C}^{(\mathbb{N})} := \{z \mid z_i = 0, \text{ for all but finite } i\}.$$

Among many choices of X, we will take

$$X[\eta] := \{(z_i)_i \mid \sum_i z_i x^i/i! \text{ converges in a nbd of } |x| \le \eta\}$$
$$\xrightarrow{\sim} \mathcal{O}_{\mathbb{C}}(\{|x| \le \eta\}),$$

for $\eta \ge 0$. They are dual Fréchet Schwartz spaces.

local admissibility and holomorphy

Consider functions on $W \subset X$, $(\mathbb{C}^{(\mathbb{N})} \hookrightarrow X \hookrightarrow \mathbb{C}^{\mathbb{N}})$.
There is a classical notion of infinite dimensional holomorphy.

Fact

- *The holomorphy is stable under several operations.*
- *The holomorphy does not in general admit our chain rules, nor the local characterization by the expansions* $\sum_{\alpha \in \mathbb{N}^{\oplus \mathbb{N}}} a_\alpha (z - \dot{z})^\alpha$.
- *For such properties, we introduced a sufficient condition: a notion of local admissibility. But its stability is not well-studied.*
- *In* $X[\eta]$*, the holomorphy and the local admissibility are equivalent.*

Therefore, by working on $X[\eta]$, we can enjoy all: the stability, the chain rules, and the local characterization property.

formal vector field D

Consider the formal vector field

$$D = \partial_x + \sum_{i\in\mathbb{N}} z_{i+1}\partial_{z_i}.$$

Then, $D[f]$ for $f(x, z_0, \ldots, z_k)$ (of finite variables) is well-defined, and the usual chain rule of differentiations for

$$g(x) := f(x, u(x), \partial_x u(x), \ldots, \partial_x^k u(x))$$

reads

$$\partial_x g(x) = D[f](x, u(x), \partial_x u(x), \ldots, \partial_x^{k+1} u(x)), \quad \text{and,}$$
$$\partial_x^m g(x) = D^m[f](x, u(x), \partial_x u(x), \ldots, \partial_x^{k+m} u(x)), \quad m \in \mathbb{N}.$$

chain rules

Holomorphic functions on $\mathbb{C}_x \times X[\eta]$ admit a similar result.

Theorem (chain rule I on $\mathbb{C}_x \times X[\eta]$)

Let $f(x,z)$ be a holomorphic function on an open set U in $\mathbb{C}_x \times X[\eta]$.

(1) *$D[f]$ converges absolutely and defines a holomorphic function on U.*

(2) *Let $u(x)$ be a holomorphic function on $\Omega \subset \mathbb{C}_x$ satisfying $\vec{u}(x) := (x, (\partial_x^i u(x))_{i\in\mathbb{N}}) \in U$ for any $x \in \Omega$. Then the composition*

$$g(x) := f \circ \vec{u}(x) = f(x, (\partial_x^i u(x))_{i\in\mathbb{N}})$$

becomes holomorphic on Ω and satisfies

$$\partial_x^m g(x) = (D^m f) \circ \vec{u}(x) = D^m[f](x, (\partial_x^i u(x))_{i\in\mathbb{N}}), \quad m \in \mathbb{N}, x \in \Omega.$$

(We omit the chain rule II.)

coupling equation for equations of normal form

Recall that we want to solve the initial value problem of a coupling equation

$$\begin{cases} \partial_t \phi + \sum_{m\in\mathbb{N}} D^m[F]\cdot\partial_{z_m}\phi = G(t,x,\phi,D[\phi]), \\ \phi(0,x,z) = z_0. \end{cases}$$

$F(t,x,z_0,z_1),\ G(t,x,z_0,z_1) \in C^0([0,r_0];\mathcal{O}(K_0))$,
$K_0 \subset \mathbb{C}^3_{(x,z_0,z_1)}$: a closed polydisc centered at the origin.

We transform it into the integral equation:

$$\begin{aligned} \phi &= T[\phi] \\ &:= z_0 - \int_0^t \sum_m D^m[F]\cdot\partial_{z_m}\phi\Big|_{t=\tau} d\tau + \int_0^t G(t,x,\phi,D[\phi])\Big|_{t=\tau} d\tau \end{aligned}$$

and shall solve it by applying the Banach fixed-point theorem to T and some suitable function spaces $\mathscr{Z}_{d,c}(\alpha,\beta)$ defined below.

domain $\Omega_{d,c,+0}$ and weight function $\omega_{d,c,0}(t,x,z)$

We define suitable weight functions and domains

$$\omega_{d,c,0} : \mathbb{R}_{\geq 0} \times \mathbb{C} \times \mathbb{C}^{\mathbb{N}} \to \mathbb{R} \sqcup \{-\infty\},$$
$$\Omega_{d,c,+0} \subset \mathbb{R}_t \times \mathbb{C}_x \times \mathbb{C}_z^{\mathbb{N}},$$

indexed by some 1-Lipschitz continuous functions $d(x)$ on $\mathbb{C}_x$ and constants $c \geq 1$.

Fact

- $d(x) := R - |x|$ *is a typical example for* $d(x)$.
- $\{0 \leq ct/r < d(x)\} \cap \{z = 0\} \subset \Omega_{d,c,+0} \subset \{0 \leq ct/r < d(x)\}$.
- *The section* $\{(x,z) \mid (s,x,z) \in \Omega_{d,c,+0}\}$ *of* $\Omega_{d,c,+0}$ *at* $t = s$ *is an open set in* $\mathbb{C}_x \times X[cs/r]$. *(Topologies depend on* s.*)*

function spaces $\mathscr{X}_{d,c}$ and $\mathscr{X}_{d,c}(\alpha, \beta)$ I

A Banach space $\mathscr{X}_{d,c}$ of functions on $\Omega_{d,c,+0}$ is defined as follows: $\phi(t,x,z) \in C^0\mathcal{O}(\Omega_{d,c,+0})$ belongs to $\mathscr{X}_{d,c}$ if there exists C such that

$$|\phi| \le C\rho, \quad |D[\phi]| \le C\rho, \quad |D^2[\phi]| \le C\rho/\omega_{d,c,0}^{1/2},$$

$$|\partial_{z_m}\phi| \le \frac{C}{\omega_{d,c,0}^{1/2}} \cdot \frac{(ct/r)^m}{m!}, \quad |\partial_{z_m} D[\phi]| \le \frac{C}{\omega_{d,c,0}^{1/2}} \sum_{i=0}^{\min\{m,1\}} \frac{(ct/r)^{m-i}}{(m-i)!},$$

for $(t,x,z) \in \Omega_{d,c,+0}$ and $m \in \mathbb{N}$.
These inequalities define the 5 semi-norms $\|\phi\|_{d,c,1}, \|\phi\|_{d,c,2}, \dots, \|\phi\|_{d,c,5}$, and the norm of $\mathscr{X}_{d,c}$ is given by

$$\|\phi\|_{d,c,\mathrm{A}} := \max\{\|\phi\|_{d,c,1}, \|\phi\|_{d,c,2}, \|\phi\|_{d,c,3}, \|\phi\|_{d,c,4}, \|\phi\|_{d,c,5}\}.$$

function spaces $\mathscr{Z}_{d,c}$ and $\mathscr{Z}_{d,c}(\alpha, \beta)$ II

We also use

$$\|\phi\|_{d,c,1\text{-}3} := \max\{\|\phi\|_{d,c,1}, \|\phi\|_{d,c,2}, \|\phi\|_{d,c,3}\},$$
$$\|\phi\|_{d,c,45} := \max\{\|\phi\|_{d,c,4}, \|\phi\|_{d,c,5}\},$$

and define complete metric spaces

$$\mathscr{Z}_{d,c}(\alpha, \beta) := \{\phi \in \mathscr{Z}_{d,c} \mid \|\phi\|_{d,c,1\text{-}3} \leq \alpha,\ \|\phi\|_{d,c,45} \leq \beta\}$$

for positive constants α, β.

Note that the factor $\omega_{d,c,0}(t,x,z)^{-1/2}$ appearing in the definition of $\mathscr{Z}_{d,c}$ plays a role as a weight function of Nagumo type.

unique solvability of coupling equation

Under a suitable choice of $d(x)$, α and β, the integral equation

$$\phi = T[\phi]$$
$$:= z_0 - \int_0^t \sum_m D^m[F] \cdot \partial_{z_m}\phi\Big|_{t=\tau} d\tau + \int_0^t G(t,x,\phi,D[\phi])\Big|_{t=\tau} d\tau$$

is equivalent to the initial value problem for $\phi \in \mathscr{Z}_{d,c}(\alpha,\beta)$, and we have

Theorem (unique solvability)

For a sufficiently large c, the integral operator T becomes a contraction map of $\mathscr{Z}_{d,c}(\alpha,\beta)$. As a conclusion, the initial value problem has a unique solution in $\mathscr{Z}_{d,c}(\alpha,\beta)$. This solution also belongs to $C^1\mathcal{O}(\Omega_{d,c,+0})$.

Roughly speaking, $C^1\mathcal{O}(\Omega_{d,c,+0})$ is

$$\{\phi \in C^0\mathcal{O}(\Omega_{d,c,+0}) \mid \partial_t\phi \in C^0\mathcal{O}(\Omega_{d,c,+0})\}.$$

heuristic calculations

We use the conventions:

$$\vec{u} = (t, x, (\partial_x^i u)_{i\in\mathbb{N}}) \quad \text{for } u(t,x),$$
$$\vec{\phi} = (t, x, (D^i[\phi])_{i\in\mathbb{N}}) \quad \text{for } \phi(t,x,z).$$

The definition of $u \mapsto \Phi[u]$ associated with ϕ and the chain rule imply

$$\Phi[u] = \phi \circ \vec{u}, \quad \text{and} \quad \overrightarrow{\Phi[u]} = \vec{\phi} \circ \vec{u}.$$

Also consider Ψ associated with ψ. Then, $\Psi \circ \Phi$ corresponds to $\psi \circ \vec{\phi}$.

$$\begin{array}{ccccc} u & \xmapsto{\Phi} & \Phi[u] & \xmapsto{\Psi} & \Psi \circ \Phi[u] \\ \vec{\bullet}\big\downarrow & & \vec{\bullet}\big\downarrow & & \vec{\bullet}\big\downarrow \\ \vec{u} & \xmapsto{\vec{\phi}\circ\bullet} & \vec{\phi} \circ \vec{u} & \xmapsto{\vec{\psi}\circ\bullet} & \vec{\psi} \circ \vec{\phi} \circ \vec{u} \end{array}$$

compositions of coupling maps

Problem

Let $d_1(x)$ and c_1 be given. Can we compose two elements $\phi, \psi \in \mathscr{Z}_{d_1,c_1}$ into $\psi \circ \vec{\phi} \in \mathscr{Z}_{d,c}$ with some $d(x)$ and c?

a (not satisfactory) answer

Take sufficiently small $d(x)$, large c, and small α_2. If $\phi, \psi \in \mathscr{Z}_{d_1,c_1}$ and $\|\phi\|_{d_1,c_1,\text{1-3}} \leq \alpha_2$, then the composition $\psi \circ \vec{\phi}$ is well-defined on $\Omega_{d,c,+0}$, belongs to $\mathscr{Z}_{d,c}$, and satisfies

$$\|\psi \circ \vec{\phi}\|_{d,c,\text{1-3}} \leq \|\psi\|_{d_1,c_1,\text{1-3}},$$
$$\|\psi \circ \vec{\phi}\|_{d,c,45} \leq C\|\psi\|_{d_1,c_1,45}\|\phi\|_{d_1,c_1,45},$$

with some constant C depending on $d_1(x)$, c_1, $d(x)$, c.

compositions of coupling equations

In this part, we refer to the initial value problem

$$\begin{cases} \partial_t\phi + \sum_{m\in\mathbb{N}} D^m[F]\cdot\partial_{z_m}\phi = G\circ\vec{\phi}, \\ \phi\big|_{t=0} = z_0, \end{cases}$$

as the *problem* $(F \to G)$.

Consider $(F \to G)$ and $(G \to H)$, take suitable $d_1(x)$, c_1, α_1, β_1, and denote by $\phi, \psi \in \mathscr{Z}_{d_1,c_1}(\alpha_1, \beta_1)$ their unique solutions respectively. Then, for the *composed problem* $(F \to H)$, we give

Theorem (composition of solutions)

Under a suitable choice of $d(x)$, c, α, β, the solutions ϕ and ψ can be composed into the unique solution $\psi\circ\vec{\phi} \in \mathscr{Z}_{d,c}(\alpha, \beta)$ to $(F \to H)$.

reversibility

Consider $(F \to G)$ and *its reversed problem* $(G \to F)$.

Let $\phi, \psi \in \mathscr{Z}_{d_1,c_1}(\alpha_1, \beta_1)$ be the unique solutions to $(F \to G)$ and to $(G \to F)$ respectively, with some $d_1(x)$, c_1, α_1, β_1.

Corollary

Under a suitable choice of $d(x)$, c, α, β, *the compositions* $\psi \circ \vec{\phi}$ *and* $\phi \circ \vec{\psi}$ *are well-defined in* $\mathscr{Z}_{d,c}(\alpha, \beta)$, *and satisfy*

$$\psi \circ \vec{\phi}(t, x, z) = \phi \circ \vec{\psi}(t, x, z) = z_0,$$

that is, $\vec{\psi} \circ \vec{\phi} = \vec{\phi} \circ \vec{\psi} = \mathrm{id}_{\Omega_{d,c,+0}}$.

Thank you for your attention.

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