Hypergeometric functions with several variabls

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Analytic, Algebraic and Geometric Aspects of Differential Equations

The Mathematical Research and Conference Center at Będlewo, Poland

14–19 September, 2014

§ Introduction

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2. Bessel, Whittaker (Confluence)
   Irregular Singularity (unramified, ramified)

3. Generalized hypergeometric, Jordan-Pochhammer
   rigid: Even family,...

4. Heun (accessory parameters)
   ($\to$ Painlevé eq.)
   non-rigid: Garnier,...

5. Appell's hypergeometric
   Aomoto-Gelfand
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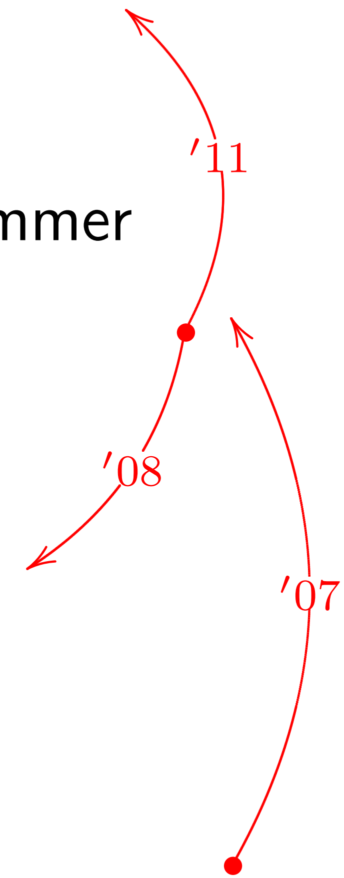

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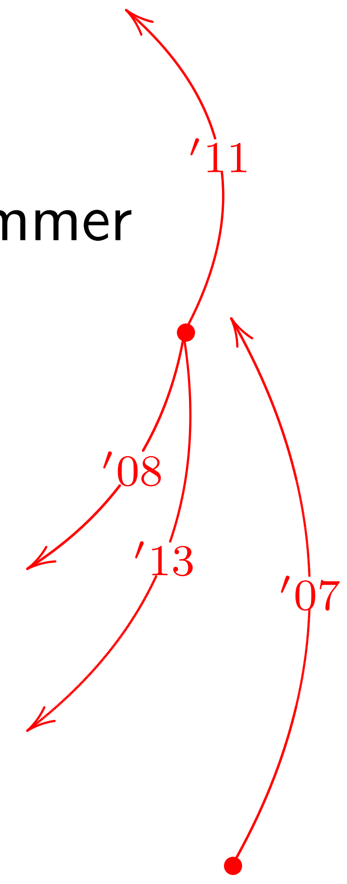

§ Gauss Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n \quad \big((a)_n=a(a+1)\cdots(a+n-1)\big)$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 s^{\beta-1}(1-xs)^{-\alpha}(1-s)^{\gamma-\beta-1}ds$$

$$=\frac{\Gamma(\gamma)x^{1-\gamma}}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^x t^{\beta-1}(1-t)^{-\alpha}(x-t)^{\gamma-\beta-1}dt \quad (t=xs)$$

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$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)}x^{1-\gamma}I_0^{\gamma-\beta}\,x^{\beta-1}(1-x)^{-\alpha}, \quad (1-x)^{-\alpha}=\sum_{n=0}^{\infty}\binom{-\alpha}{n}x^n$$

$$I_0^{\mu}v(x):=\frac{1}{\Gamma(\mu)}\int_0^x u(t)(x-t)^{\mu-1}dt \quad (=\partial^{-\mu}v(x))$$

$$I_0^{\mu}x^{\lambda}=\frac{1}{\Gamma(\mu)}\int_0^x t^{\gamma}(x-t)^{\mu-1}dt$$

$$=\frac{x^{\gamma+\mu}}{\Gamma(\mu)}\int_0^1 s^{\gamma}(1-s)^{\mu-1}ds=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}x^{\gamma+\mu}$$

§ Addition and Middle Convolution in Weyl algebra

$W[x] := \mathbb{C}[x,\partial] \qquad (\partial = \frac{d}{dx},\ [\partial, x] = 1$: Weyl algebra$)$

$W(x) := \mathbb{C}(x) \otimes_{\mathbb{C}[x]} W(x)$

$v(x) \mapsto v_1(x) = \varphi(x)v(x), \quad v(x) \mapsto v_2(x) = I_0^{\mu} v(x) = \partial^{-\mu} v(x)$

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$x \mapsto \partial^{-\mu} \circ x \circ \partial^\mu = x - \mu\partial^{-1}, \quad \vartheta \mapsto \vartheta - \mu, \quad \vartheta := x\partial$

- $P_2 = \mathrm{RAd}(\partial^{-\mu})P = mc_\mu(P) := \partial^{-m}\,\mathrm{Ad}(\partial^{-\mu})(\partial^k P) \in W[x]$

$mc_\mu \circ mc_{\mu'} = mc_{\mu+\mu'}, \quad mc_0 = \mathrm{id}$ (middle convolution)

$\partial^k P = \sum_{i\ge 0,\ j\ge 0} c_{i,j}\partial^i\vartheta^j \xrightarrow{\mathrm{Ad}(\partial^{-\mu})} \sum_{i\ge 0,\ j\ge 0} c_{i,j}\partial^i(\vartheta-\mu)^j \in W[x]$

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$$\mathrm{RAd}(x^{\gamma-1}) \circ \mathrm{RAd}(\partial^{\beta-\gamma}) \circ \mathrm{RAd}(x^{\beta-1}(1-x)^{-\alpha})\partial$$
$$= x(1-x)\partial^2 + \big(\gamma - (\alpha+\beta+1)x\big)\partial - \alpha\beta$$

§ Fuchsian ordinary differential equations

- $\forall$ solution $u(x)$ of linear ODE $Pu = 0$ on $\mathbb{P}^1_{\mathbb{C}}$ satisfies

$$|u(x)| < \exists C|x|^{-\exists m} \quad (|\arg x| : \text{bounded and } x \to 0)$$

at $\forall$ singular point $c_j = 0$ $(j = 0, 1, \ldots, p$ by linear frac. transf.)

$$\Leftrightarrow P = \Big(\prod_{j=0}^{p-1}(x - c_j)^n\Big)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_1(x)\partial + a_0(x) \in W[x],\ \partial := \tfrac{d}{dx}$$

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$u_{j,i} \sim x^{\lambda_{j,i}} \log^{k_{j,i}} x$ $(i = 1, \ldots, n)$, $\qquad \{(\lambda_{j,i}, k_{j,i})\,;\, i = 1, \ldots, n\}$

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- Analyze additions and middle conv.

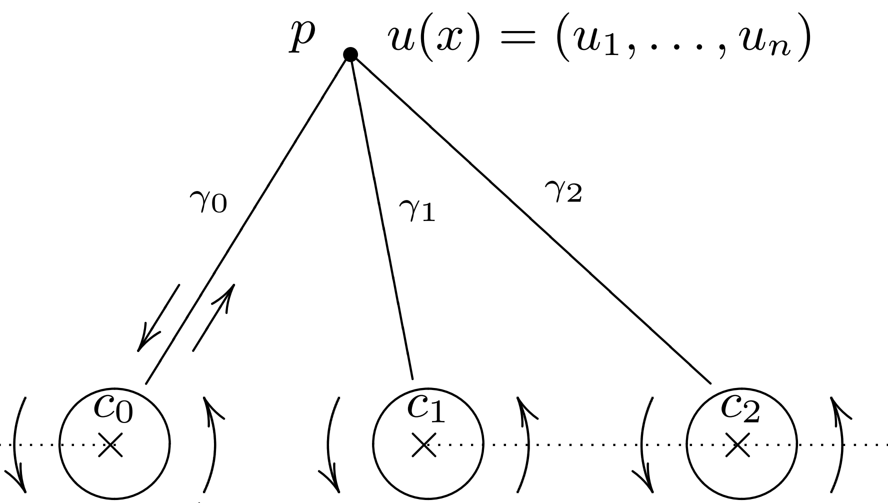
on the space of Fuchsian ODE's!

- (Local) monodromy:

$\gamma_j u(z) = u(z) M_j \ (M_j \in GL(n, \mathbb{C}))$

$M_j = G_j e^{2\pi\sqrt{-1}A_j} G_j^{-1} \sim e^{2\pi\sqrt{-1}A_j}$

M_j : semisimple $\Rightarrow A_j = \mathrm{diag}(\lambda_{j,1}, \dots \lambda_{j,n})$



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Multiplicities of eigenvalues of M_j $\Rightarrow$ partitions of n

$n = \operatorname{ord}\mathbf{m} = m_{j,1} + \cdots + m_{j,n_j}$ $(j = 0, \ldots, p)$: spectral type at c_j

char. exponents at c_j : $\{\lambda_{j,\nu} + i \mid \nu = 1, \ldots, n_j,\ 0 \le i < m_{j,\nu}\}$

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Definition. P has the generalized Riemann scheme (GRS)

$$\{\lambda_{\mathbf{m}}\} := \begin{Bmatrix} x = c_0 = \infty & c_1 & \cdots & c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix} \quad [\mu]_{(m)} := \begin{pmatrix} \mu \\ \mu+1 \\ \vdots \\ \mu+m-1 \end{pmatrix}$$

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: $(p+1)$-tuples of partitions of $n = \operatorname{ord}\mathbf{m}$ (spectral type)

Semisimple local monodromy if $\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z}$

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Fuchs condition (FC): $|\{\lambda_{\mathbf{m}}\}| := \sum m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m} + \frac{1}{2}\operatorname{idx}\mathbf{m} = 0$

$\operatorname{idx}\mathbf{m} := \sum_{j,\nu} m_{j,\nu}^2 - (p-1)(\operatorname{ord}\mathbf{m})^2$ (index of rigidity, Katz)

$$Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\} \quad \underset{mc_{-\mu}}{\overset{mc_{\mu}}{\rightleftarrows}} \quad P'u = 0 \text{ with } \{\lambda'_{\mathbf{m}'}\} \qquad (P \neq \partial)$$

We may assume $\lambda_{j,1} = 0 \ \ (j = 1, \ldots, p)$, $\lambda_{0,1} = 1 + \mu$

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$$m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1} \cdot d(\mathbf{m}),\ d(\mathbf{m}) = m_{0,1} + \cdots + m_{p,1} - (p-1) \operatorname{ord} \mathbf{m}$$

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$d(\mathbf{m})$: maximal value $\Leftarrow$ $m_{j,1} \geq m_{j,\nu}$ by addition and μ

($\mathbf{m}$ is monotone $\overset{\text{def}}{\Leftrightarrow} m_{j,1} \geq m_{j,2} \geq \cdots$)

Fact $\quad d(\mathbf{m}) \cdot \operatorname{ord} \mathbf{m} = \operatorname{idx} \mathbf{m} + \sum_{j,\nu} (m_{j,1} - m_{j,\nu}) m_{j,\nu}$

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- $\mathrm{idx}\,\mathbf{m} = 2 \Rightarrow d(\mathbf{m}) > 0$ reduced to trivial eq. $\partial u = 0$
- $\mathbf{m}$ is basic $\overset{\text{def}}{\Leftrightarrow}$ $d(\mathbf{m}) \leq 0$, $\mathbf{m}$ is monotone and irreducibly realizable

$\mathbf{m}$ is basic and $\mathrm{idx}\,\mathbf{m} = 0 \Rightarrow d(\mathbf{m}) = 0$ and $m_{j,1} = m_{j,\nu}$

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\begin{array}{ccc}
\{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\} \\
\downarrow u \mapsto \begin{cases} \partial^{-\mu} u \\ (x - c_j)^{\lambda_j} u \end{cases} & \circlearrowright & \downarrow \begin{cases} s_\alpha \ (\alpha \in \Pi) : \text{ reflections} \\ +\lambda_j \Lambda^0_{0,j} \end{cases} \\
\{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\}
\end{array}
$$

Theorem. For any ι, $\exists$ only finite $\#$ of basic $\mathbf{m}$ with $\operatorname{idx}\mathbf{m} = \iota$.

- $\operatorname{idx}\mathbf{m} = 0 \Rightarrow$ basic $\mathbf{m}$: $\tilde{D}_4$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{E}_9$ (4 cases) by Kostov
- The irreducibly realizable $\mathbf{m}$ and mc_μ with additions correspond to the positive roots $\alpha_{\mathbf{m}}$ of a star-shaped Kac-Moody root system and the actions by its Weyl group (by Crawley-Boevey)

$\mathbf{m}$ is rigid $\Leftrightarrow$ $\alpha_{\mathbf{m}}$ is a real root, $\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$

$$\begin{array}{ccc}
\{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\} \\
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\end{array}$$

Theorem. $\forall$ Irreducibly realizable $\mathbf{m}$, $\exists$(univeral model)$P(\lambda, g) \in \mathbb{C}[\lambda] \otimes \mathbb{C}[g] \otimes W[x]$ such that $P(\lambda, g)u = 0$ has $\{\lambda_{\mathbf{m}}\}$ with (FC).

Any irred. $Qv = 0$ with $\{\lambda_{\mathbf{m}}\}$ is a specialization of $P(\lambda, g)u = 0$.

Here $g = (g_1, \ldots, g_N)$ with $N = 1 - \frac{1}{2}\operatorname{idx}\mathbf{m}$. (rigid $\Leftrightarrow$ $\operatorname{idx}\mathbf{m} = 2$)

§ Fractional calculus of Weyl algebra

Unified and computable interpretation (⇒ a computer program) of

- Existence and Construction of equations (⇒ classical limit)
- Integral representation of solutions
- Series expansion of solutions
- Reducibility of the monodromy
- Contiguity relations
- Polynomial solutions
- Connection problem
- Confluences/Unfoldings and Irregular Singularities
- Several variables (PDE)   (Appell HG. etc)

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(Connection formula) If $m_{1,n_1} = 1$ and $n_1 > 1$ and $n_2 > 1$, then

$$\frac{c'(\lambda'_{1,n_1} \rightsquigarrow \lambda'_{2,n_2})}{\Gamma(\lambda'_{1,n_1} - \lambda'_{1,1} + 1)\Gamma(\lambda'_{2,2} - \lambda'_{2,n_2})} = \frac{c(\lambda_{1,n_1} \rightsquigarrow \lambda_{2,n_2})}{\Gamma(\lambda_{1,n_1} - \lambda_{1,1} + 1)\Gamma(\lambda_{2,1} - \lambda_{2,n_2})}.$$

§ Gauss hypergeometric function

$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$: Singularities: $\{0,1,\infty\} \subset \mathbb{P}^1_{\mathbb{C}}$

$$\left\{\begin{matrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & a & ;x \\ 1-c & c-a-b & b & \end{matrix}\right\}$$

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$$\mathbb{P}^1_{\mathbb{C}} \supset \{t_0,\, t_1,\, t_\infty,\, t_x\} \xrightarrow{\text{linear frac. transf.}} \{0,\, 1,\, \infty,\, x\}$$

$$x = \frac{(t_x - t_0)(t_\infty - t_1)}{(t_1 - t_0)(t_\infty - t_x)}$$

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$$\mathbb{P}^1_{\mathbb{C}}\supset\{t_0,\,t_1,\,t_\infty,\,t_x\}\mapsto$$

$$\left(\frac{(t_x-t_0)(t_\infty-t_1)}{(t_1-t_0)(t_\infty-t_x)}\right)^d\cdot\left(\frac{(t_1-t_x)(t_\infty-t_0)}{(t_1-t_0)(t_\infty-t_x)}\right)^e F(a,b,c;\tfrac{t_x-t_0}{t_1-t_0}\tfrac{t_\infty-t_1}{t_\infty-t_x})$$

$$\left\{\begin{matrix} t_x=t_0 & t_x=t_1 & t_x=t_\infty & t_0=t_1 & t_0=t_\infty & t_1=t_\infty \\ 0+d & 0+e & a-d-e & a-d-e & e & d \\ 1-c+d & c-a-b+e & b-d-e & b-d-e & c+e-a-b & 1-c+d \end{matrix}\right\}$$

Fuchsian ODE with $k+3$ singular points $\in \mathbb{P}^1_{\mathbb{C}}$

$$\Rightarrow \ \{\text{Singular points}\} \xrightarrow{\text{lin. frac. transf.}} \{0, 1, \infty, y_1, \ldots, y_k\}$$

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Solutions of rigid Fuchsian ODE have integral representations

$\rightarrow$ Functions with variables $(x, y_1, \ldots, y_k)$

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$\rightarrow$ {Hypergeometric functions with $(k+1)$-variables $(x, y_1, \ldots, y_k)$}

$\rightarrow$ {Hypergeometric systems on the moduli space of $k+4$ poins in $\mathbb{P}^1_{\mathbb{C}}$}

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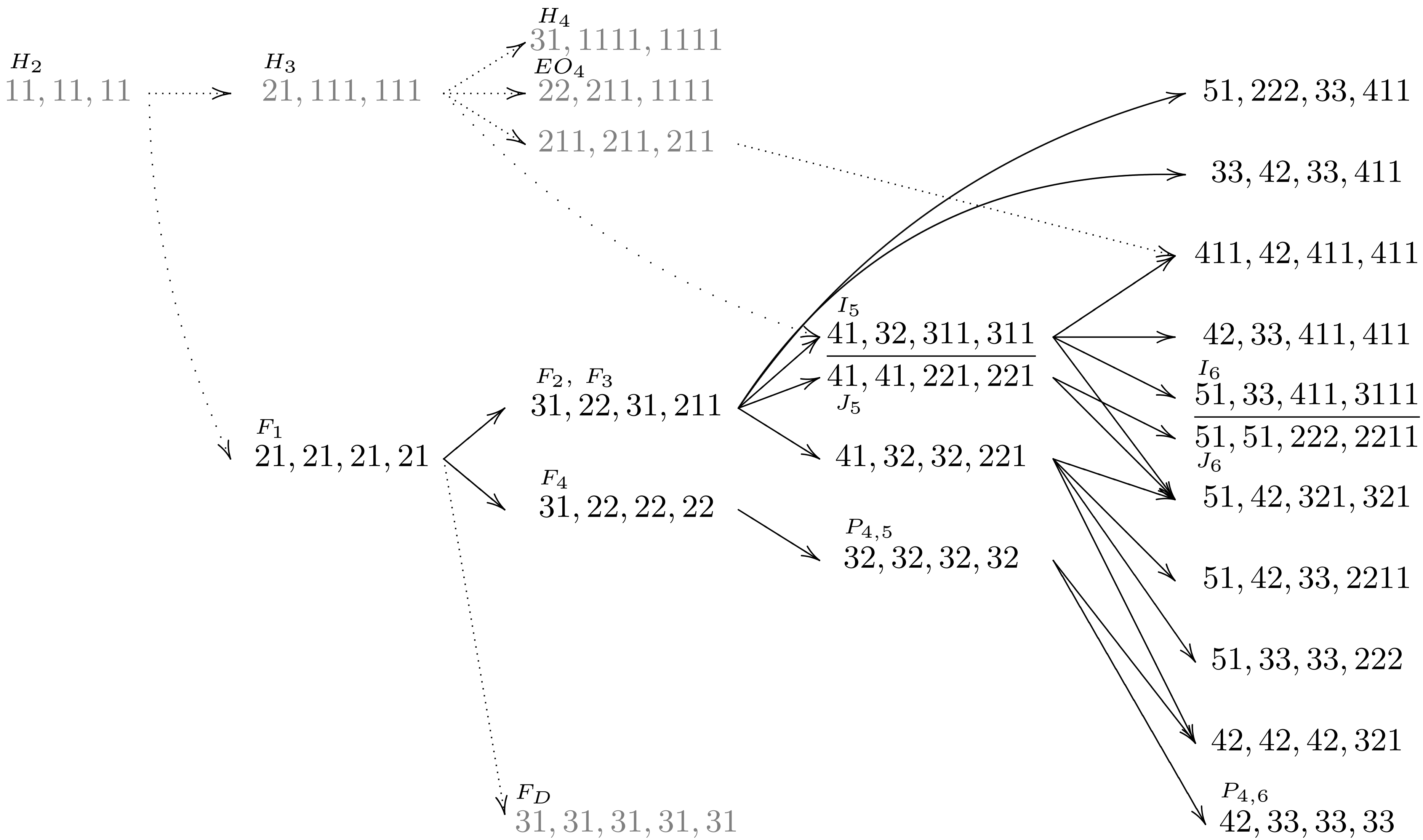
Rigid ODE

Order	2	3	4	5	6	7	8	9	10	11	12
1 variable	1	1	3	5	13	20	45	74	142	212	421
2 variables		1	2	4	11	16	35	58	109	156	299
3 variables			1	1	3	5	12	17	43	52	104
4 variables				1	0	1	3	5	8	14	24

Hierarchy of rigid quartets



$$\text{rank} \Leftrightarrow \text{ord}, \quad \#\text{parameters} = \sum \big(\#\text{ blocks at sing. point} - 1\big)$$

§ Additions and Middle Convolutions

$$du = \left(A_1 \frac{dx}{x} + A_2 \frac{d(x-y)}{x-y} + A_3 \frac{d(x-1)}{x-1} + A_4 \frac{dy}{y} + A_5 \frac{d(y-1)}{y-1} \right) u$$

$$\mathrm{Ad}_\lambda : (A_1,\ A_2,\ A_3,\ A_4,\ A_5) \mapsto (A_1+\lambda_1,\ A_2+\lambda_2,\ A_3+\lambda_3,\ A_4,\ A_5)$$

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$$\tilde{A}_1 = \begin{pmatrix} A_1 + \mu & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ A_1 & A_2 + \mu & A_3 \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1 & A_2 & A_3 + \mu \end{pmatrix},$$

$$\tilde{A}_4 = \begin{pmatrix} A_4 + A_2 & -A_2 & 0 \\ -A_1 & A_4 + A_1 & 0 \\ 0 & 0 & A_4 \end{pmatrix}, \ \tilde{A}_5 = \begin{pmatrix} A_5 & 0 & 0 \\ 0 & A_5 + A_3 & -A_3 \\ 0 & -A_2 & A_5 + A_2 \end{pmatrix}$$ (by Haraoka)

$$\mathrm{mc}_\mu(A_j) := \tilde{A}_j \mod \begin{pmatrix} \ker A_1 \\ \ker A_2 \\ \ker A_3 \end{pmatrix} \oplus \ker(\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3)$$ (Dettweiler-Reiter)

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$$\tilde{A}_1 = \begin{pmatrix} A_1+\mu & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \tilde{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ A_1 & A_2+\mu & A_3 \\ 0 & 0 & 0 \end{pmatrix},\ \tilde{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1 & A_2 & A_3+\mu \end{pmatrix},$$

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Reduction of $\mathbf{m}$: Arrange $\mathbf{m}$ to be monotone by adjacent transpositions

$\leftrightarrows$ $m_{j,1} \mapsto m_{j,1} - (m_{0,1} + \cdots + m_{p,1} - (p-1)\,\mathrm{ord}\,\mathbf{m})$ (middle convolution)

§ Additions and Middle Convolutions

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$$\mathrm{Ad}_\lambda:(A_1,\ A_2,\ A_3,\ A_4,\ A_5)\mapsto(A_1+\lambda_1,\ A_2+\lambda_2,\ A_3+\lambda_3,\ A_4,\ A_5)$$

$$\tilde A_1=\begin{pmatrix}A_1+\mu & A_2 & A_3\\ 0&0&0\\ 0&0&0\end{pmatrix},\ \tilde A_2=\begin{pmatrix}0&0&0\\ A_1 & A_2+\mu & A_3\\ 0&0&0\end{pmatrix},\ \tilde A_3=\begin{pmatrix}0&0&0\\ 0&0&0\\ A_1&A_2&A_3+\mu\end{pmatrix},$$

$$\tilde A_4=\begin{pmatrix}A_4+A_2 & -A_2 & 0\\ -A_1 & A_4+A_1 & 0\\ 0&0&A_4\end{pmatrix},\ \tilde A_5=\begin{pmatrix}A_5&0&0\\ 0&A_5+A_3&-A_3\\ 0&-A_2&A_5+A_2\end{pmatrix}$$ (by Haraoka)

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$$F_4:\ \underline{3}1,\underline{2}2,\underline{2}2,\underline{2}2\ \underset{1}{\overset{-1}{\to}}\ 21,\underline{12},12,12\ \underset{1}{\to}\ 21,21,\underline{12},12\ \underset{1}{\to}\ 21,21,21,\underline{12}\ \underset{1}{\to}\ \underline{2}1,\underline{2}1,\underline{2}1,\underline{2}1(F_1)$$

$$\underset{2}{\overset{-2}{\to}}\ \underline{01},01,01,01\ \underset{1}{\to}\ 10,\underline{01},01,01\ \underset{1}{\to}\ 10,10,\underline{01},01\ \underset{1}{\to}\ 10,10,10,\underline{01}\ \underset{1}{\to}\ 10,10,10,10$$

$8+1$ steps, $3+2+2+2-(4-2)\times 4=1$, $2+2+2+2-(4-2)\times 3=2$

§ Examples (two variables, order ≤ 6)

Jordan-Pochhammer ($\Rightarrow F_1$) $\underline{21, 21, 21, 21}$: order 3 with 4 parameters ($1^4 \cdot 2^1$)

$$21, 21, 21, 21 = 10, 10, 10, 01 \oplus 11, 11, 11, 20 \quad (4) \qquad \text{(reducibility)}$$

$$= 2(10, 10, 10, 10) \oplus 01, 01, 01, 01 \quad (1)$$

$|\{\lambda_{\mathbf{m}_\beta}\}| \notin \mathbb{Z}$ $(\forall \mathbf{m} = k\mathbf{m}_\beta \oplus \mathbf{m}_\gamma) \Leftrightarrow$ irreducible (ODE)

$$\begin{Bmatrix} x=0 & x=1 & x=y & x=\infty \\ [0]_2 & [0]_2 & [0]_2 & [e]_2 \\ a & b & c & d \end{Bmatrix}$$

$a + b + c + d + 2e = 2, 0$ (FC)

$10, 10, 10, 01 : d \notin \mathbb{Z}$ ($c + e, \ldots$ 4 cond.)

$10, 10, 10, 10 : e \notin \mathbb{Z}$ (Cond. for irred.)

§ Examples (two variables, order ≤ 6)

Jordan-Pochhammer ($\Rightarrow F_1$) $\underline{21,21,21,21}$: order 3 with 4 parameters ($1^4 \cdot 2^1$)

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$10,10,10,\underline{10}$

$$0{+}0{+}0{+}0{-}(4{-}2)\cdot 1 = -2$$

$$\underset{1}{\leftarrow} 10,10,\underline{10},01 \underset{1}{\leftarrow} 10,\underline{10},01,01 \underset{1}{\leftarrow} \underline{10},01,01,01 \underset{1}{\leftarrow} \underline{0}1,\underline{0}1,\underline{0}1,\underline{0}1 \overset{-2}{\underset{2}{\leftarrow}} \underline{2}1,\underline{2}1,\underline{2}1,\underline{2}1$$

$$* \overset{+2}{\to} 2(10,10,10,10)$$

$$-1{+}0{+}0{+}0{-}(4{-}2)\cdot 0 = -1$$

$$* \to \underline{-1}1,\underline{0}0,\underline{0}0,\underline{0}0 \overset{+1}{\to} 01,10,10,10$$

$$* \to \underline{00},-11,00,00 \to \underline{0}0,\underline{-1}1,\underline{0}0,\underline{0}0 \overset{+1}{\to} 10,01,10,10$$

$$* \to 00,\underline{00},-11,00 \to \underline{00},00,-11,00 \to \underline{0}0,\underline{0}0,\underline{-1}1,\underline{0}0 \overset{+1}{\to} 10,10,01,10$$

$$* \to 00,00,\underline{00},-11 \to 00,\underline{00},00,-11 \to \underline{00},00,00,-11 \to \underline{0}0,\underline{0}0,\underline{0}0,\underline{-1}1 \overset{+1}{\to} 10,10,10,01$$

§ Examples (two variables, order ≤ 6)

F_1 $\underline{21, 21, 21, 21}$: order 3 with 4 parameters ($1^4 \cdot 2^1$)

$21, 21, 21, 21 \ \to \ 01, 01, 01, 01 \quad H_2 : 11, 11, 11, 20$ (by a middle convolution)

$21, 21, 21, 21 = 10, 10, 10, 01 \oplus 11, 11, 11, 20 \quad (4)$ (reducibility)

$= 2(10, 10, 10, 10) \oplus 01, 01, 01, 01 \quad (1)$

$$\left\{\begin{matrix} x=0 & x=1 & x=y & x=\infty & y=0 & y=1 & y=\infty \\ [0]_2 & [0]_2 & [0]_2 & [e]_2 & [0]_2 & [0]_2 & [-c-e]_2 \\ a & b & c & d & a+c+2e & b+c+2e & d \end{matrix}\right\}$$

$a + b + c + d + 2e = 2, 0$ (FC)

$10, 10, 10, 01 : d \notin \mathbb{Z}$ ($c + e, \ldots$ 4 cond.)

$10, 10, 10, 10 : e \notin \mathbb{Z}$

$10, 10, 01, 10 : c + e \quad 10, 01, 10, 10 : b + e \quad 10, 01, 10, 10 : a + e \notin \mathbb{Z}$ (Cond for irred.)

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		21	21	21	21	2
t_0	21		21	21	21	2
t_y	21	21		21	21	2
t_1	21	21	21		21	2
t_x	21	21	21	21		2

$F_2,\ F_3 \quad \underline{211, 22, 31, 31}$: order 4 with 5 parameters $(1^6 \cdot 2^2)$

$$
\begin{aligned}
211,22,31,31 &\ \rightarrow\ F_1 : 201,21,21,21 \quad H_2 : 011,02,11,11 \\
211,22,31,31 &= 010,01,10,10 \oplus 201,21,21,21 \quad (4) \\
&= 101,11,11,20 \oplus 110,11,20,11 \quad (2) \\
&= 2(100,01,10,10) \oplus 011,20,11,11 \quad (2)
\end{aligned}
$$

$$
\begin{Bmatrix}
x=0 & x=y & x=1 & x=\infty & y=0 & y=1 & y=\infty & x=y=\infty \\
[0]_2 & [0]_3 & [0]_3 & [d]_2 & [0]_3 & [0]_2 & [-a-b-e-d]_2 & [f]_3 \\
[a]_2 & b & c & e & a+b+2d & [e-f]_2 & f & -a-b-e \\
 & & & f & & & a-f &
\end{Bmatrix}
$$

$$2a+b+c+2d+e+f=0,\ 3$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		211	211	211	211	-8
t_0	211		31	31	22	2
t_y	211	31		22	31	2
t_1	211	31	22		31	2
t_x	211	22	31	31		2

$F_4 \quad 22, 22, \underline{31}, 22$: order 4 with 4 parameters $(1^8 \cdot 2^1)$

$$22, 22, 31, 22 \ \to \ F_1 : 12, 12, 21, 12$$
$$22, 22, 31, 22 = 01, 01, 10, 01 \oplus 21, 21, 21, 21 \quad (8)$$
$$= 2(11, 11, 20, 11) \oplus 00, 00, (-1)1, 00 \quad (1)$$

$$\left\{\begin{matrix} x=0 & x=1 & x=y & y=0 & y=1 & x=\infty & y=\infty & x=y=\infty \\ [b]_2 & [c]_2 & [0]_3 & [-b]_2 & [-c]_2 & [d]_2 & [b+c+d]_2 & b+c+2d \\ [0]_2 & [0]_2 & a & [0]_2 & [0]_2 & [e]_2 & [b+c+e]_2 & b+c+2e \\ & & & & & & & [0]_2 \end{matrix}\right\}$$

$$a + 2b + 2c + 2d + 2e = 0, \ 3$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		211	22	211	22	-4
t_0	211		22	211	22	-4
t_y	22	22		22	31	2
t_1	211	211	22		22	-4
t_x	22	22	31	22		2

Above cond. : $u(\alpha, \beta, \gamma; x, y) = F(\alpha, \beta, \gamma; x) \cdot F(\alpha, \beta, \gamma; y)$

I_5 $\overline{41}, \underline{32}, 311, 311,$ J_5 $\overline{41, 41}, 221, 221$: order 5 with 6 parameters $(1^6 \cdot 2^4)$

$$41, 41, 221, 221 \;\rightarrow\; F_2 : 31, 31, 121, 220 \quad F_1 : 21, 21, 021, 021$$

$$41, 32, 311, 311 \;\rightarrow\; F_2 : 31, 22, 211, 301 \quad H_3 : 21, 30, 111, 111 \quad H_2 : 11, 02, 011, 011$$

$$\begin{aligned} 41, 41, 221, 221 &= 10, 10, 001, 010 \oplus 31, 31, 220, 211 \quad (4) \\ &= 20, 11, 110, 110 \oplus 21, 30, 111, 111 \quad (2) \\ &= 2(10, 10, 100, 100) \oplus 21, 21, 021, 021 \quad (4) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		311	311	221	221	-10
t_0	311		32	311	41	2
t_y	311	32		311	41	2
t_1	221	311	311		221	-10
t_x	221	41	41	221		2

$\underline{41}, 32, 32, 221$: order 5 with 5 parameters $(1^7 \cdot 2^3)$

$$41,32,32,221 \ \rightarrow \ 31,22,22,220 \quad F_2 : 31,22,31,121 \quad F_1 : 21,12,12,021$$

$$\begin{aligned} 41,32,32,221 &= 10,10,10,001 \oplus 31,22,22,220 \quad (1) \\ &= 10,01,10,010 \oplus 31,31,22,211 \quad (4) \\ &= 20,11,11,101 \oplus 21,21,21,120 \quad (2) \\ &= 2(10,10,10,100) \oplus 21,12,12,021 \quad (2) \\ &= 2(20,11,11,110) \oplus 01,10,10,001 \quad (1) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		311	32	2111	32	-6
t_0	311		32	2111	32	-6
t_y	32	32		221	41	2
t_1	2111	2111	221		221	-18
t_x	32	32	41	221		2

$P_{4,5}$ $32, 32, 32, 32$: order 5 with 4 parameters $(1^8 \cdot 2^2)$

$$32, 32, 32, 32 \ \rightarrow \ F_4 : 22, 22, 22, 31 \quad F_1 : 12, 12, 12, 12$$

$$\begin{aligned} 32, 32, 32, 32 &= 10, 10, 10, 01 \oplus 22, 22, 22, 31 \quad (4) \\ &= 21, 21, 21, 12 \oplus 11, 11, 11, 20 \quad (4) \\ &= 2(10, 10, 10, 10) \oplus 12, 12, 12, 12 \quad (1) \\ &= 2(21, 21, 21, 21) \oplus -(10, 10, 10, 10) \quad (1) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		221	221	211	32	-10
t_0	221		221	221	32	-10
t_y	221	221		221	33	-10
t_1	221	221	221		32	-10
t_x	32	32	32	32		2

$I_6\ \overline{51}, 411, \underline{33}, 3111, \quad J_6\ \overline{51, 51}, 222, 2211$: order 6 with 7 parameters

$$51, 411, 33, 3111 \ \rightarrow \ 41, 311, 23, 3011 \ \ 21, 111, 03, 0111 \ (1^{12} \cdot 3^2)$$

$$\begin{aligned} 51, 411, 33, 3111 &= 10, 100, 10, 0100 \oplus 41, 311, 23, 3011 \quad (6) \\ &= 20, 110, 11, 1100 \oplus 31, 301, 22, 2011 \quad (6) \\ &= 3(10, 100, 10, 1000) \oplus 21, 111, 03, 0111 \quad (2) \end{aligned}$$

$$51, 51, 222, 2211 \ \rightarrow \ 41, 41, 122, 2201 \ \ 31, 31, 022, 0211 \ (1^8 \cdot 2^6)$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		222	2211	3111	3111	-26
t_0	222		222	411	411	-12
t_y	2211	222		51	51	2
t_1	3111	411	51		33	2
t_x	3111	411	51	33		2

$\underline{42}, 411, 411, 411$: order 6 with 7 parameters $(1^{10} \cdot 2^1 \cdot 4^1)$

$$42, 411, 411, 411 \;\rightarrow\; 32, 311, 311, 401 \quad 40, 211, 211, 211 \quad 02, 011, 011, 011$$

$$\begin{aligned} 42, 411, 411, 411 &= 10, 100, 100, 010 \oplus 32, 311, 311, 401 \quad (6) \\ &= 21, 210, 210, 210 \oplus 21, 201, 201, 201 \quad (4) \\ &= 2(01, 100, 100, 100) \oplus 40, 211, 211, 211 \quad (1) \\ &= 4(10, 100, 100, 100) \oplus 02, 011, 011, 011 \quad (1) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		222	411	222	411	-12
t_0	222		411	222	411	-12
t_y	411	411		411	42	2
t_1	222	222	411		411	-12
t_x	411	411	42	411		2

$\underline{51}, 42, 321, 321$: order 6 with 6 parameters $\quad (1^7 \cdot 2^3 \cdot 3^1)$

$51,42,321,321 \;\rightarrow\; 41,32,221,320 \;\; 41,32,311,311 \;\; 41,41,221,221 \;\; 31,22,121,301 \;\; 21,12,021,021$

$$
\begin{aligned}
51,42,321,321 &= 10,10,100,001 \oplus 41,32,221,320 \quad (2)\\
&= 10,10,010,010 \oplus 41,32,311,311 \quad (1)\\
&= 10,01,100,100 \oplus 41,41,221,221 \quad (1)\\
&= 20,11,110,101 \oplus 31,31,211,220 \quad (2)\\
&= 30,21,111,111 \oplus 21,21,210,210 \quad (1)\\
&= 2(10,10,010,100) \oplus 31,31,220,211 \quad (2)\\
&= 2(20,11,110,110) \oplus 11,20,101,101 \quad (1)\\
&= 3(10,10,100,100) \oplus 21,12,021,021 \quad (1)
\end{aligned}
$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		2211	321	3111	321	-22
t_0	2211		321	3111	321	-22
t_y	321	321		42	51	2
t_1	3111	3111	42		42	-8
t_x	321	321	51	42		2

$42,42,3111,3111 \rightarrow 22,22,1111,1111 \;(-8)$

$(42,3111,42,3111 : 42,321,51,321) \;\mapsto\; (22,1111,22,1111 : 22,22,4,22)$

$\underline{51}, 411, 33, 222$: order 6 with 6 parameters $(1^6 \cdot 2^6)$

$$51, 411, 33, 222 \ \rightarrow \ 31, 211, 13, 022$$

$$51, 411, 33, 222 = 20, 101, 11, 011 \oplus 31, 310, 22, 211 \quad (6)$$

$$= 2(10, 100, 10, 100) \oplus 31, 211, 13, 022 \quad (6)$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		3111	222	21111	411	-22
t_0	3111		33	3111	33	-12
t_y	222	33		411	51	2
t_1	21111	3111	411		222	-22
t_x	411	33	51	222		2

$\underline{51}, 42, 33, 2211$: order 6 with 6 parameters $(1^8 \cdot 2^5)$

$$51, 42, 33, 2211 \;\to\; 41, 32, 23, 2201 \quad 31, 22, 13, 0211$$

$$\begin{aligned} 51, 42, 33, 2211 &= 10, 10, 10, 0010 \oplus 41, 32, 23, 2201 \quad (4) \\ &= 20, 11, 11, 1010 \oplus 31, 31, 22, 1201 \quad (4) \\ &= 2(10, 10, 10, 1000) \oplus 31, 22, 13, 0211 \quad (4) \\ &= 2(10, 11, 11, 1100) \oplus 11, 20, 11, 0011 \quad (1) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		21111	2211	21111	2211	-36
t_0	21111		42	411	42	-6
t_y	2211	42		33	51	2
t_1	21111	411	33		33	-10
t_x	2211	42	51	33		2

$$42, 42, 411, 21111 \;\to\; 22, 22, 211, 1111,\; 32, 32, 311, 2111 \; (-6)$$
$$(42, 411, 42, 21111 : 42, 33, 51, 2211) \mapsto (22, 211, 22, 1111 : 22, 22, 40, 22)$$
$$(42, 411, 42, 21111 : 42, 33, 51, 2211) \mapsto (32, 311, 32, 2111 : 32, 32, 41, 221)$$

$42, 411, 411, \underline{33}$: order 6 with 6 parameters $(1^{10} \cdot 3^2)$

$$42, 411, 411, 33 \ \rightarrow \ 41, 311, 311, 23 \ \ 12, 111, 111, 03$$

$$\begin{aligned} 42, 411, 411, 33 &= 01, 100, 100, 10 \oplus 41, 311, 311, 23 \quad (2) \\ &= 11, 110, 200, 11 \oplus 31, 301, 211, 22 \quad (4) \\ &= 21, 210, 201, 12 \oplus 21, 201, 210, 21 \quad (4) \\ &= 3(10, 100, 100, 10) \oplus 12, 111, 111, 03 \quad (2) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		2211	42	2211	42	-12
t_0	2211		411	411	411	-8
t_y	42	411		411	33	2
t_1	2211	411	411		411	-8
t_x	42	411	33	411		2

$$411, 411, 411, 2211 \ \rightarrow \ 211, 211, 211, 211, \ \ 311, 311, 311, 221 \ (-8)$$

$$(411, 411, 2211, 411 : 411, 411, 42, 33) \mapsto (211, 211, 211, 211 : 211, 31, 31, 22) \ F_2$$

$51, 33, 33, 222$: order 6 with 5 parameters $(1^{12} \cdot 2^3)$

$$51, 33, 33, 222 \ \rightarrow \ 41, 23, 23, 122$$

$$51, 33, 33, 222 = 10, 10, 10, 100 \oplus 41, 23, 23, 122 \quad (12)$$

$$= 2(20, 11, 11, 11) \oplus 11, 11, 11, 02 \quad (3)$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		21111	222	21111	222	-32
t_0	21111		33	3111	33	-14
t_y	222	33		33	51	2
t_1	21111	3111	33		33	-14
t_x	222	33	51	33		2

$42, 411, 33, 33$: order 6 with 5 parameters $(1^6 \cdot 2^5)$

$$42, 411, 33, 33 \ \rightarrow \ 22, 211, 13, 13$$

$$\begin{aligned} 42, 411, 33, 33 &= 20, 110, 11, 11 \oplus 22, 301, 22, 22 \quad (2) \\ &= 21, 210, 21, 12 \oplus 21, 201, 12, 21 \quad (4) \\ &= 2(10, 100, 10, 10) \oplus 22, 211, 13, 13 \quad (4) \\ &= 2(11, 200, 11, 11) \oplus 20, 011, 11, 11 \quad (1) \end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		2211	222	2211	411	-22
t_0	2211		33	3111	33	-16
t_y	222	411		411	42	-4
t_1	2211	3111	411		33	-16
t_x	411	33	42	33		2

$$42, 411, 411, 222 \ \rightarrow \ 22, 211, 211, 22 \ (-4)$$

$$(42, 411, 41, 222 : 42, 411, 33, 33) \mapsto (22, 22, 211, 211 : 22, 30, 22, 22)$$

$42, 42, 42, 321$: order 6 with 5 parameters $(1^8 \cdot 2^2 \cdot 3^1)$

$$42,42,42,321 \ \rightarrow \ 32,32,32,320 \quad 32,32,41,221 \quad 22,22,22,301 \quad 12,12,12,021$$

$$\begin{aligned}
42,42,42,321 &= 10,10,10,001 \oplus 32,32,32,320 \quad (1) \\
&= 10,10,01,100 \oplus 32,32,41,221 \quad (3) \\
&= 11,11,20,110 \oplus 31,31,22,211 \quad (3) \\
&= 21,21,21,200 \oplus 21,21,21,120 \quad (1) \\
&= 2(10,10,10,010) \oplus 22,22,22,301 \quad (1) \\
&= 2(21,21,21,210) \oplus 00,00,00,(-1)01 \quad (1) \\
&= 3(10,10,10,100) \oplus 12,12,12,021 \quad (1)
\end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		2211	2211	2211	321	-28
t_0	2211		321	321	42	-14
t_y	2211	321		321	42	-14
t_1	2211	321	321		42	-14
t_x	321	42	42	42		2

$P_{4,6}$ $42,33,33,33$: order 6 with 4 parameters $(1^{12}\cdot 2^2)$

$$42,33,33,33 \;\to\; 32,32,32,32$$

$$\begin{aligned}42,33,33,33 &= 10,10,10,10 \oplus 32,23,23,23 \quad (8)\\ &= 21,21,21,21 \oplus 21,12,12,12 \quad (4)\\ &= 2(20,11,11,11) \oplus 02,11,11,11 \quad (1)\\ &= 2(31,22,22,22) \oplus -(20,11,11,11) \quad (1)\end{aligned}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		2211	222	2211	33	-22
t_0	2211		222	2211	33	-22
t_y	222	222		222	42	-16
t_1	2211	2211	222		33	-22
t_x	33	33	42	33		2

$$\mathbf{m}_\alpha = k \cdot \mathbf{m}_\beta \oplus \mathbf{m}_\gamma$$ (Give condition for reducibility of ODE)

- $\operatorname{ord} \mathbf{m}_\gamma > 0 \ \Rightarrow$ Gives reducibility $\quad (|\{\lambda_{\mathbf{m}_\beta}\}| \notin \mathbb{Z})$
- $\operatorname{ord} \mathbf{m}_\gamma < 0 \ \Rightarrow \ k > 1$ and $\mathbf{m}_\alpha = (k^2 - 2)(-\mathbf{m}_\gamma) \oplus \mathbf{m}_\delta$

  $\Rightarrow$ not necessary (contained in other conditions)
- $\operatorname{ord} \mathbf{m}_\gamma = 0 \ \Rightarrow \ k > 1$ and

  $$\mathbf{m}_\gamma = 0 \cdots 0, \cdots, \underline{0 \cdots (-1) 0 \cdots 1 0 \cdots 0}, 0 \cdots, 0 \cdots$$

  $\Rightarrow$ reeducible but product form?

$\mathbf{m}_\alpha = k \cdot \mathbf{m}_\beta \oplus \mathbf{m}_\gamma$ (Give condition for reducibility of ODE)

- $\operatorname{ord} \mathbf{m}_\gamma > 0 \ \Rightarrow$ Gives reducibility $\quad (|\{\lambda_{\mathbf{m}_\beta}\}| \notin \mathbb{Z})$
- $\operatorname{ord} \mathbf{m}_\gamma < 0 \ \Rightarrow \ k > 1$ and $\mathbf{m}_\alpha = (k^2-2)(-\mathbf{m}_\gamma) \oplus \mathbf{m}_\delta$

  $\Rightarrow$ not necessary (contained in other conditions)
- $\operatorname{ord} \mathbf{m}_\gamma = 0 \ \Rightarrow \ k > 1$ and

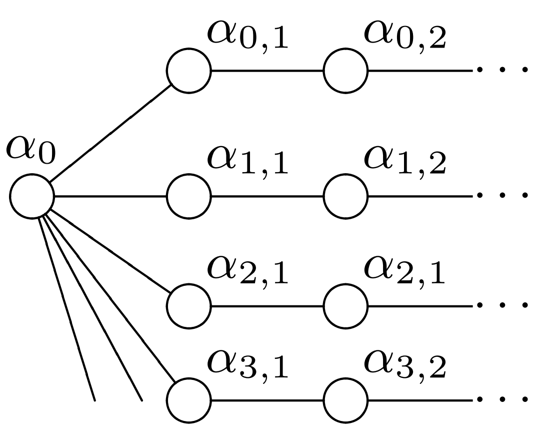
  $\mathbf{m}_\gamma = 0\cdots0, \cdots, \underline{0\cdots(-1)0\cdots10\cdots0}, 0\cdots, 0\cdots$

  $\Rightarrow$ product form?

$n_{j,\nu} := m_{j,\nu+1} + m_{j,\nu+2} + \cdots, \quad n := n_{j,0}$

$\alpha_{\mathbf{m}} := n\alpha_0 + \sum_{j,\,\nu} n_{j,\nu}\alpha_{j,\nu}$

$\alpha = \alpha_{\mathbf{m}} \Leftrightarrow \mathbf{m} = \mathbf{m}_\alpha$



$$r_i(x) := x - 2\frac{(x|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i \quad (x \in \sum \mathbb{C}\alpha_i,\ i \in I)$$

$W := \langle r_i \mid i \in I \rangle$: Weyl group, $\quad I = \{0, (j,\nu) \mid j \geq 0,\ \nu \geq 1\}$

$\alpha_0 = w_{\mathbf{m}}\alpha_{\mathbf{m}} \quad (\exists_1 w_{\mathbf{m}} \in W$ with minimal length)

$\Delta(\mathbf{m}) := \Delta_+^{re} \cap w_{\mathbf{m}}^{-1}\Delta_-^{re} = \{\beta\}, \quad k = (\alpha_{\mathbf{m}}|\beta)$

§ Transformations

- addition

  $u(x,y) \mapsto \phi(x)^{\lambda} u(x,y),\ \ \phi(x) = x$ or $(1-x)$ or $(y_i - x)$

- middle convolution

  $u(x,y) \mapsto \int_{c_j}^{x} u(t,y)(x-t)^{\mu-1} dt$

- coordinate transformations and/or symmetry

  1. $x \mapsto 1-x$
  2. $x \mapsto \frac{1}{x}$
  3. $(x, \cdots, y_i, y_{i+1}, \cdots) \mapsto (x, \cdots, y_{i+1}, y_i, \cdots)$
  4. $(x, y_1, \cdots, y_k) \mapsto (y_1, x, y_2, \cdots, y_k)$
  5. $(x, y_1, \ldots, y_k) \mapsto (x, \frac{x}{y_1}, \frac{x}{y_2}, \cdots, \frac{x}{y_k})$

     $$A_{x=0} \leftrightarrow A_{x=y_1=\cdots=y_k=0}, \qquad A_{y_i=1} \leftrightarrow A_{x=y_i},$$

     $$A_{x=\infty} \leftrightarrow A_{x=y_1=\cdots=y_k=\infty}, \qquad A_{y_i=0} \leftrightarrow A_{y_i=\infty}$$

  This Group of symmetry $\simeq \mathfrak{S}_{k+4}$

  $$k=0 \Rightarrow |\mathfrak{S}_{0+4}| = 24, \qquad k=1 \Rightarrow |\mathfrak{S}_{1+4}| = 120$$

§ Power series

$$u(x,y) = x^{\lambda_{0,n_1}}(1-x)^{\lambda(0)_{1,max}}(1-y)^{\lambda(0)_{2,max}} \sum_{\nu_{1,1}=1}^{\infty}\sum_{\nu_{2,1}=1}^{\infty}\cdots\sum_{\nu_{1,K}=1}^{\infty}\sum_{\nu_{2,K}=1}^{\infty}$$

$$\prod_{i=0}^{K-1}\frac{\bigl(\lambda(i)_{0,n_1}-\lambda(i)_{0,max}+1\bigr)_{\sum_{s=1}^{2}\sum_{t=i+1}^{K}\nu_{s,t}}}{\bigl(\lambda(i)_{0,n_1}-\lambda(i)_{0,max}+\mu(i)+1\bigr)_{\sum_{s=1}^{2}\sum_{t=i+1}^{K}\nu_{s,t}}}$$

$$\cdot\prod_{i=1}^{K}\prod_{s=1}^{2}\frac{\bigl(\lambda(i-1)_{s,max}-\lambda(i)_{s,max}\bigr)_{\nu_{s,i}}}{\nu_{s,i}!}\cdot x^{\sum_{i=1}^{K}\nu_{1,i}}\cdot y^{\sum_{i=1}^{K}\nu_{2,i}}$$

$$\lambda(i-1)_{s,max} = \lambda(i)_{s,max} \;\Rightarrow\; \nu_{s,i} = 0$$

- One time additions at $x = y$

  $\Rightarrow$ Coeffients of $x^i y^j$ are quotients of products of Γ functions

$${}_mF_{m-1} : H_m = 1^m, (m-1)1, 1^m \xleftarrow{H_1} H_{m-1}$$

$$I_{2m} \xleftarrow{mH_1} H_m, \quad I_{2m+1} \xleftarrow{(m+1)H_1} H_m$$

$$1,1,1 \to\; 11,\underline{2}1,11 \to\; 111,\underline{3}1,111 \to\; 1111,\underline{4}1,1111 \to \cdots$$

$$1^m, \underline{1}^m, \underline{(m-1)}1 \to I_{2m} : m1^m, (m+1)1^{m-1}, \underline{(2m-1)}1, mm$$

Risa/Asir

```
Gauss=[[alpha,beta],[1-gamma,0],[0,gamma-alpha-beta]]; /* Gauss */
os_md.getbygrs(Gauss,["All","dviout", "operator"])$

os_md.getbygrs([[1,1,1],[1,1,1],[2,1]],["All","dviout","operator"])$
 /* 3F2 */

os_md.shiftop("11,11,11","1-1,00,00"|zero=1,dviout=1,all=1)$

os_md.m2mc("21,21,21,21","All")$    /* F1 */
```

Riemann Scheme

$$P\begin{Bmatrix} x=\infty & 0 & 1 & \\ \alpha & -\gamma+1 & 0 & ;x \\ \beta & 0 & -\alpha-\beta+\gamma & \end{Bmatrix} \tag{1}$$

Connection formula

$$c(0:0\rightsquigarrow 1:-\alpha-\beta+\gamma)=\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \tag{2}$$

Recurrence relation shifting the last exponents at ∞, 0, 1

$$u_{0,0,0}-u_{+1,0,-1}=\frac{(\gamma)}{(\alpha)}u_{0,+1,-1} \tag{3}$$

Integral representation

$$\begin{aligned}&\int_p^x x^{-\gamma+1}(x-s_0)^{-\alpha+\gamma-1}s_0^{\alpha-1}(1-s_0)^{-\beta}ds_0\\ &\sim\frac{\Gamma(-\alpha+\gamma)\Gamma(\alpha)}{\Gamma(\gamma)}x^0\quad(p=0,\ x\to 0)\end{aligned} \tag{4}$$

Series expansion

$$\sum_{n\geq 0} \frac{(\beta)_n(\alpha)_n}{(\gamma)_n n!} x^n \tag{5}$$

Irreduciblity $\Leftrightarrow$ any value of the following linear forms $\notin \mathbb{Z}$

$$\begin{aligned} &\beta \quad \alpha \\ &\beta-\gamma \quad \alpha-\gamma \end{aligned} \tag{6}$$

which cooorespond to the decompositions

$$\begin{aligned} 11,11,11 &= 10,10,01 \oplus 01,01,10 \\ &= 10,01,10 \oplus 01,10,01 \\ &= 01,10,10 \oplus 10,01,01 \\ &= 10,10,10 \oplus 01,01,01 \end{aligned} \tag{7}$$

Operator

$$-x(x-1)\partial^2 - ((\alpha+\beta+1)x-\gamma)\partial - \beta\alpha \tag{8}$$

Riemann Scheme

$$P\begin{Bmatrix} x=\infty & 0 & 1 & \\ a_0 & b_0 & [0]_{(2)} & ;x \\ a_1 & b_1 & c & \\ a_2 & 0 & & \end{Bmatrix} \tag{9}$$

Fuchs condition

$$c+b_1+b_0+a_2+a_1+a_0-2 \tag{10}$$

Connection formula

$$c(0:0 \rightsquigarrow 1:c) = \frac{\Gamma(-b_0+1)\Gamma(-b_1+1)\Gamma(-c)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} \tag{11}$$

Recurrence relation shifting the last exponents at ∞, 0, 1

$$u_{0,0,0} - u_{+1,0,-1} = \frac{(-b_0+1)(-b_1+1)}{(a_0)(a_1)} u_{0,+1,-1} \tag{12}$$

Integral representation

$$\int_p^x \int_p^{s_0} x^{b_0}(x-s_0)^{-b_0-a_0} s_0^{b_1+a_0-1}(s_0-s_1)^{-b_1-a_1} s_1^{a_1-1}(1-s_1)^{-a_2} ds_1 ds_0$$
$$\sim \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(-b_0-a_0+1)\Gamma(-b_1-a_1+1)}{\Gamma(-b_0+1)\Gamma(-b_1+1)} x^0 \quad (p=0,\ x\to 0) \tag{13}$$

Series expansion

$$\sum_{n\geq 0} \frac{(a_0)_n (a_1)_n (a_2)_n}{(-b_0+1)_n (-b_1+1)_n n!} x^n \tag{14}$$

Irreduciblity $\Leftrightarrow$ any value of the following linear forms $\notin \mathbb{Z}$

$$\begin{array}{llllll} a_0 & a_1 & a_2 & & & \\ b_0+a_0 & b_0+a_1 & b_0+a_2 & b_1+a_0 & b_1+a_1 & b_1+a_2 \end{array} \tag{15}$$

which cooorespond to the decompositions

$$
\begin{aligned}
111,111,21 &= 110,110,11 \oplus 001,001,10 \\
&= 010,001,10 \oplus 101,110,11 \\
&= 001,010,10 \oplus 110,101,11 \\
&= 010,010,10 \oplus 101,101,11 \\
&= 100,001,10 \oplus 011,110,11 \\
&= 100,010,10 \oplus 011,101,11 \\
&= 001,100,10 \oplus 110,011,11 \\
&= 010,100,10 \oplus 101,011,11 \\
&= 100,100,10 \oplus 011,011,11
\end{aligned} \tag{16}
$$

Operator

$$
\begin{aligned}
&x^2(x-1)\partial^3 + x((a_2+a_1+a_0+3)x+b_1+b_0-3)\partial^2 \\
&+(((a_1+a_0+1)a_2+(a_0+1)a_1+a_0+1)x+(-b_0+1)b_1+b_0-1)\partial+a_0a_1a_2
\end{aligned} \tag{17}
$$

Shift Operator

$$
\begin{aligned}
&\left\{\begin{matrix} x=\infty & 0 & 1 \\ a & 0 & 0 \\ -a-b-c+1 & b & c \end{matrix}\right\} = \{u \mid Pu = 0\} \\
&\underset{Q_2}{\overset{Q_1}{\rightleftarrows}} \left\{\begin{matrix} x=\infty & 0 & 1 \\ a+1 & 0 & 0 \\ -a-b-c & b & c \end{matrix}\right\} \\
Q_1 &= (2a+b+c)x(x-1)\partial + a((2a+b+c)x-a-c) \\
Q_2 &= -(2a+b+c)x(x-1)\partial + (a+b+c)((2a+b+c)x-a-b) \\
Q_2Q_1 &\equiv a(a+c)(a+b)(a+b+c) \mod W(x)P \\
P &= -x(x-1)\partial^2 + ((b+c-2)x-b+1)\partial + a(a+b+c-1)
\end{aligned} \tag{18}
$$

Riemann scheme

$$\begin{Bmatrix} x=0 & x=y & x=1 & y=0 & y=1 & x=\infty & y=\infty & x=y=\infty \\ [0]_2 & [0]_2 & [0]_2 & [0]_2 & [0]_2 & [d]_2 & [-b-d]_2 & [-a-b-c-2d]_2 \\ a & b & c & a+b+2d & b+c+2d & -a-b-c-2d & -a-b-c-2d & -b \end{Bmatrix} \tag{19}$$

Spectral types : 21,21,21,21 : 21,21,21,21

By the decompositions

$$\begin{aligned} 21,21,21,21 &= 10,10,10,01 \oplus 11,11,11,20 \\ &= 10,10,01,10 \oplus 11,11,20,11 \\ &= 10,01,10,10 \oplus 11,20,11,11 \\ &= 01,10,10,10 \oplus 20,11,11,11 \\ &= 2(10,10,10,10) \oplus 01,01,01,01 \end{aligned} \tag{20}$$

irreducibility $\Leftrightarrow$ $\emptyset = \mathbb{Z} \cap$

$$[d, c+d, b+d, a+d, a+b+c+2d] \tag{21}$$

The equation in a Pfaff form is

$$
\begin{aligned}
du = \Bigg(&\begin{pmatrix} a & b+d & c+d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{dx}{x} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ a+d & b & c+d \\ 0 & 0 & 0 \end{pmatrix} \frac{d(x-y)}{x-y} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a+d & b+d & c \end{pmatrix} \frac{d(x-1)}{x-1} \\
&+ \begin{pmatrix} b+d & -(b+d) & 0 \\ -(a+d) & a+d & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{dy}{y} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & c+d & -(c+d) \\ 0 & -(b+d) & b+d \end{pmatrix} \frac{d(y-1)}{y-1} \Bigg) u
\end{aligned} \tag{22}
$$

Thank you!   End!

T. Oshima, *Fractional calculus of Weyl algebra and Fuchsian differential equations*, MSJ Memoirs **28**(2012), xix+203pp.

T. Oshima, `ftp://akagi.ms.u-tokyo.ac.jp/pub/math/muldif`