The Stokes phenomenon in certain partial differential equations

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Abstract

We consider the Stokes phenomenon in certain PDEs. At first we introduce basic definitions related to the Gevrey asymptotics. Then we discuss the theory of Borel k-summability, which will be the fundamental tool that we use to prove the most important theorems. At the main part we focus our attention to derive the Stokes lines and anti-Stokes lines for the heat equation. Based on the obtained results we apply them for generalizations of the heat equation with meromorphic initial conditions.

The Gevrey asymptotics

Definition 1

Let S be a given sector in the complex plane and let $f \in \mathcal{O}(S)$.
A formal power series of Gevrey order s ($s \in \mathbb{R}$):
$\hat{f}(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}[[t]]_s$ is called **asymptotic expansion of order** $\boldsymbol{s}$ of f in S, if:

$$\forall_{S^* \prec S} \ \exists_{A,B<\infty} \ \forall_{N \in \mathbb{N}_0} \ \forall_{t \in S^*} \ |f(t) - \sum_{n=0}^{N} a_n t^n| \leqslant AB(N!)^s |t|^{N+1},$$

where S^* is a proper subsector of S.
If this is so, we will use notation: $f(t) \sim_s \hat{f}(t)$ in S.

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- Theorem 1. *Ritt's theorem for Gevrey asymptotics*
  Let $\hat{x}(t) \in \mathbb{C}[[t]]_s$, where $s > 0$. Let S be a sector of opening α, where $0 < \alpha \leqslant s\pi$. Then:

$$\exists_{x(t) \in \mathcal{O}(S)} \text{ such, that} : \forall_{t \in S} \; x(t) \sim_s \hat{x}(t) \text{ in } S.$$

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$$\exists_{x(t)\in\mathcal{O}(S)} \text{ such, that} : \forall_{t\in S}\; x(t) \sim_s \hat{x}(t) \text{ in } S.$$

- Theorem 2. *Watson's lemma*
  Let S be a sector of opening α such that $\alpha > s\pi$ and $s > 0$. Suppose that $x(t) \in \mathcal{O}(S)$ satisfy $x(t) \sim_s 0$ in S. Then:

$$\forall_{t\in S}\; x(t) \equiv 0.$$

Modified k-summability method in a direction d

The title method consist in transforming a formal power series $\hat{x}(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}[[t]]_{1/k}$ into a holomorphic function. First, we apply **the Borel modified transform of order k** $(k > 0)$ defined by:

$$(\breve{B}_k \hat{x})(t) := \sum_{n=0}^{\infty} \frac{a_n\, t^n\, n!}{\Gamma\left(1 + n(1 + \frac{1}{k})\right)},$$

then we use **the Ecalle's operator** i.e:

$$(E_{k,d}\, g)(t) := t^{-k/(1+k)} \int_0^{\infty e^{id}} g(s)\, C_{(k+1)/k}\left((s/t)^{\frac{k}{1+k}}\right) ds^{\frac{k}{1+k}},$$

where $k > 0, d \in \mathbb{R}, g(s) = (\breve{B}_k \hat{x})(s)$ and for $\alpha > 1$:

$$C_\alpha(\tau) := \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!\, \Gamma\left(1 - \frac{n+1}{\alpha}\right)}.$$

Modified k-summability method in a direction d

Finally, we obtain so-called k-sum of $\hat{x}(t)$ given by:
$x(t) = (E_{k,d}g)(t) \in \mathcal{O}(S(d, \epsilon + \pi/k, r))$, where $S(d, \epsilon + \pi/k, r)$ is a bounded sector.

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- We can use modified k-summability method in a direction d if the following conditions are satisfied:
  1. $\hat{x}(t) \in \mathbb{C}[[t]]_{1/k}$.
  2. $(\breve{B}_k\hat{x})(t) \in \mathcal{O}(S_d)$, where $S_d = S(d, \epsilon)$ is an unbounded sector in a direction d.
  3. $|(\breve{B}_k\hat{x})(t)| \leqslant C_1 e^{C_2|t|^k}$, for some $C_1, C_2 > 0$ as $t \longrightarrow \infty, t \in S_d$.

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  3. $|(\breve{B}_k \hat{x})(t)| \leqslant C_1 e^{C_2|t|^k}$, for some $C_1, C_2 > 0$ as $t \longrightarrow \infty, t \in S_d$.
- In this case we say that $\hat{x}(t)$ is ***k*-summable in a direction *d***
  $(k > 0, d \in \mathbb{R})$.

Modified k-summability method in a direction d

Observe that by Watson's lemma k-sum $x(t)$ is the unique holomorphic function on $S(d, \epsilon + \pi/k, r)$ satisfying:

$$x(t) \sim_{1/k} \hat{x}(t) \text{ in } S(d, \epsilon + \pi/k, r).$$

The Stokes phenomenon

Definition 2

Assume that $\hat{x}(t) \in \mathbb{C}[[t]]_{1/k}$ is k-summable in all directions $d \in (\phi - \epsilon, \phi + \epsilon)$, but singular direction $d = \phi$ (for some $\epsilon > 0$). Then **the Stokes line** is a set $\mathcal{L}_\phi = \{t \in \mathbb{C} : \arg t = \phi\}$ and **the anti-Stokes lines** are sets $\mathcal{L}_{\phi \pm \frac{\pi}{2k}} = \{\ t \in \mathbb{C} : \arg t = \phi \pm \frac{\pi}{2k}\}$.

Remark

Assume that S is a sector with a opening π/k in a direction ϕ. Let $f(t), g(t) \in \mathcal{O}(S)$ be k-sums of $\hat{x}(t)$ in directions $\phi - \epsilon/2$ and $\phi + \epsilon/2$ respectively. It means that $f(t) \sim_{1/k} \hat{x}(t)$ and $g(t) \sim_{1/k} \hat{x}(t)$ for all $t \in S$.
We write $r(t) := |f(t) - g(t)|$ for all $t \in S$. Then $r(t)$ is minimal on the Stokes line $\mathcal{L}_\phi$ and satisfies inequalities $|f(t)| \leqslant r(t)$ or $|g(t)| \leqslant r(t)$ on the anti-Stokes lines $\mathcal{L}_{\phi \pm \frac{\pi}{2k}}$.

The Stokes and anti-Stokes lines for the heat equation

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- Let us consider the heat equation : $u_t(t,z) = u_{zz}(t,z)$ with $u(0,z) = \phi(z)$, where $\phi \in \mathcal{O}(D)$. As easily seen, this Cauchy problem has the unique formal solution:

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(z)\, t^n}{n!}.$$

The Stokes and anti-Stokes lines for the heat equation

Theorem 3.(*D.A Lutz, M .Miyake and R. Schäfke 1999*)

Suppose that $\hat{u}(t,z)$ is the unique formal solution of the Cauchy problem of heat equation:

$$\begin{cases} u_t(t,z) = u_{zz}(t,z) \\ u(0,z) = \phi(z) \in \mathcal{O}^2\Big(\big(D \cup S(\frac{d}{2},\varepsilon) \cup S(\frac{d}{2}+\pi,\varepsilon)\big) \times D \Big) \end{cases}$$

Then $\hat{u}(t,z)$ is 1-summable in a direction d and the 1-sum of $\hat{u}(t,z)$ is represented by:

$$u^{\theta}(t,z) = E_{1,\theta}\, \breve{B}_1\, \hat{u}(t,z) = \frac{1}{\sqrt{4\pi t}} \int_{e^{i\frac{\theta}{2}}\mathbb{R}} \phi(z+s)\, e^{\frac{-s^2}{4t}}\, ds,$$

for small t such that $\arg t \in (-\frac{\pi}{2}+\theta; \frac{\pi}{2}+\theta)$.

The Stokes and anti-Stokes lines for the heat equation

Theorem 4. *(S.M, B.P 2015)*

Assume that $\phi(z) = \frac{a}{z-z_0} + \tilde{\phi}(z)$ and $a, z_0 \in \mathbb{C}$, $\tilde{\phi}(z) \in \mathcal{O}^2(\mathbb{C})$.
Let $\delta := 2\arg(z_0 - z)$. Then Stokes lines are: $\mathcal{L}_\delta$ and anti-Stokes lines are: $\mathcal{L}_{-\frac{\pi}{2}+\delta}$ and $\mathcal{L}_{\frac{\pi}{2}+\delta}$.
Moreover:

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = -i\sqrt{(\pi/t)}ae^{-\frac{(z_0-z)^2}{4t}},$$

for $\arg t \in (-\frac{\pi}{2} + \delta + \varepsilon; \frac{\pi}{2} + \delta - \varepsilon)$.

The idea of the proof of Theorem 4

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- $$u^{\delta+\varepsilon}(t,z)=\frac{1}{\sqrt{4\pi t}}\int_0^{e^{\frac{i(\delta+\varepsilon)}{2}}\infty}\big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\big)e^{-\tilde{s}^2/4t}\,d\tilde{s}$$

  for $\arg t\in(-\frac{\pi}{2}+\delta+\varepsilon;\frac{\pi}{2}+\delta+\varepsilon)$, also:

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- $$u^{\delta+\varepsilon}(t,z)=\frac{1}{\sqrt{4\pi t}}\int_0^{e^{\frac{i(\delta+\varepsilon)}{2}}\infty}(\phi(z+\tilde{s})+\phi(z-\tilde{s}))e^{-\tilde{s}^2/4t}\,d\tilde{s}$$

  for $\arg t\in(-\frac{\pi}{2}+\delta+\varepsilon;\frac{\pi}{2}+\delta+\varepsilon)$, also:

- $$u^{\delta-\varepsilon}(t,z)=\frac{1}{\sqrt{4\pi t}}\int_0^{e^{\frac{i(\delta-\varepsilon)}{2}}\infty}(\phi(z+\tilde{s})+\phi(z-\tilde{s}))e^{-\tilde{s}^2/4t}\,d\tilde{s}$$

  for $\arg t\in(-\frac{\pi}{2}+\delta-\varepsilon;\frac{\pi}{2}+\delta-\varepsilon)$.

The idea of the proof of Theorem 4

Hence (and based on the residue theorem) we derive:

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) =$$

$$= -2\pi i \operatorname{res}_{\tilde{s}=z_0-z}\left[\frac{1}{\sqrt{4\pi t}}\Big(\phi(z+\tilde{s}) + \phi(z-\tilde{s})\Big)e^{-\tilde{s}^2/4t}\right] =$$

$$= -i\sqrt{(\pi/t)}\lim_{\tilde{s}\to z_0-z}(\tilde{s}-(z_0-z))\left[\Big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\Big)e^{-\tilde{s}^2/4t}\right] =$$

The Stokes and anti-Stokes lines for the heat equation

The idea of the proof of Theorem 4

$$= -i\sqrt{(\pi/t)} \lim_{\tilde{s}\to z_0 - z} (\tilde{s} - (z_0 - z)) \Bigg[\Bigg(\frac{a}{\tilde{s} - (z_0 - z)} + \tilde{\phi}(z + \tilde{s}) +$$

$$+ \frac{a}{z - \tilde{s} - z_0} + \tilde{\phi}(z - \tilde{s}) \Bigg) e^{-\frac{\tilde{s}^2}{4t}} \Bigg] = -i\sqrt{(\pi/t)}\, a\, e^{-\frac{(z_0 - z)^2}{4t}},$$

for $\arg t \in (-\frac{\pi}{2} + \delta + \varepsilon; \frac{\pi}{2} + \delta - \varepsilon)$.

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The idea of the proof of Theorem 4

Notice that for such t that $\arg t \in (-\frac{\pi}{2} + \delta + \varepsilon; \frac{\pi}{2} + \delta - \varepsilon)$ occur:

$-i\sqrt{(\pi/t)}\, a\, e^{-\frac{(z_0-z)^2}{4t}} \sim_1 0$, $u^{\delta+\varepsilon} \sim_1 \hat{u}$ and $u^{\delta-\varepsilon} \sim_1 \hat{u}$.

Hence the Stokes line is: $\mathcal{L}_\delta$, the anti-Stokes lines are: $\mathcal{L}_{-\frac{\pi}{2}+\delta}$, $\mathcal{L}_{\frac{\pi}{2}+\delta}$.

□

The Stokes and anti-Stokes lines for generalizations of the heat equation

In this part we will generalize results presented in the previous slides.

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- Let us consider an equation: $\partial_t^p u(t,z) = \partial_z^q u(t,z), p, q \in \mathbb{N}$ with the following conditions: $u(0,z) = \phi(z) \in \mathcal{O}(D)$ and $\partial_t^j u(0,z) = 0$ for $j = 1, 2, \ldots, p-1$.
  The equation below has the unique formal solution represented by:

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\phi^{(qn)}(z) t^{pn}}{(pn)!}.$$

The Stokes and anti-Stokes lines for generalizations of the heat equation

Theorem 5. (*M.Miyake 1999, S.M, B.P 2015*)

Suppose that $\hat{u}(t,z)$ is the unique formal solution of:

$$\begin{cases} \partial_t^p u = \partial_z^q u, \quad 1 \leqslant p < q \\ u(0,z) = \phi(z) \in \mathcal{O}^{\frac{q}{q-p}}\Big((D \cup S(\frac{dp}{q} + \frac{2\pi l}{q}, \varepsilon)) \times D\Big), \ l = 0, \dots, q-1 \\ \partial_t^j u(0,z) = 0, \quad j = 1, 2, \dots, p-1 \end{cases}$$

The Stokes and anti-Stokes lines for generalizations of the heat equation

Theorem 5. (*M.Miyake 1999, S.M, B.P 2015*)

Then $\hat{u}(t,z)$ is $\frac{p}{q-p}$-summable in direction d, and the $\frac{p}{q-p}$-sum of $\hat{u}(t,z)$ for all $\theta \in (d-\frac{\epsilon}{2}, d+\frac{\epsilon}{2})$ is given by:

$$u^{\theta}(t,z) = E_{\frac{p}{q-p},\theta} \check{B}_{\frac{p}{q-p}} \hat{u}(t,z) =$$

$$= \frac{1}{q\sqrt[q]{t^p}} \int_0^{e^{i\theta p/q}\infty} \left(\phi(z+\tilde{s}) + \cdots + \phi(z + e^{\frac{2(q-1)\pi i}{q}}\tilde{s})\right) C_{\frac{q}{p}}(\tilde{s}/\sqrt[q]{t^p})\, d\tilde{s},$$

for small t such that $\arg t \in (-\frac{\pi(q-p)}{2p}+\theta; \frac{\pi(q-p)}{2p}+\theta)$.

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Analogously to Theorem 4, we formulate the following results:

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- Theorem 6.(*S.M, B.P 2015*)

  Assume that $\phi = \frac{a}{z-z_0} + \tilde{\phi}(z)$, where $a \in \mathbb{C}, \tilde{\phi}(z) \in \mathcal{O}^{\frac{q}{q-p}}(\mathbb{C})$ and denote $\delta := \frac{q}{p}\arg(z_0 - z)$. Then the Stokes lines are: $\mathcal{L}_\delta$ and the anti-Stokes lines are: $\mathcal{L}_{-\frac{\pi(q-p)}{2p}+\delta}$, $\mathcal{L}_{\frac{\pi(q-p)}{2p}+\delta}$.

  Moreover:

$$u^{\delta+\varepsilon} - u^{\delta-\varepsilon} = \frac{-2\pi i}{q\sqrt[q]{t^p}}\, a\, C_{\frac{q}{p}}\Big(\frac{z_0 - z}{\sqrt[q]{t^p}}\Big),$$

$$\text{for } \arg t \in \Big(-\frac{\pi(q-p)}{2p} + \delta + \varepsilon; \frac{\pi(q-p)}{2p} + \delta - \varepsilon\Big).$$

Remarks

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- It is worth pointing out that we can consider in theorems 4 and 6 function $\phi(z)$ with more than one pole and with poles of higher order i.e $\phi(z) = \sum_{l=1}^{n}\sum_{m=1}^{r}\frac{a_{lm}}{(z-z_l)^m} + \tilde{\phi}(z)$.

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- We also emphasize that presented results can be generalize to moment partial differential equations introduced by W.Balser and M.Yoshino [3].

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THANK YOU!