Multi-level asymptotics for singularly perturbed linear partial differential equations with ultraholomorphic coefficients

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Some related works in the literature

Aim: To study the effect that the properties of the coefficients of some equations have on the asymptotics or growth properties of the corresponding solutions.

- Plenty of results regard analytic (e.g. Cauchy-Kovalevski theorem) and Gevrey regularity for PDEs.
- Recent results are concerned with singular perturbation problems (Y. Sibuya (2003), M. Canalis-Durand, J. Mozo and R. Schäfke (2007), S. Malek (2012), A. Lastra, S. Malek and J.S. (2013), S. Kamimoto (2013)), where Gevrey asymptotics and $k-$summability appear.
- Only few references consider more general classes of functions, defined by estimating the growth of their $n-$derivatives in terms of a sequence of real numbers $\mathbb{M} = (M_n)_{n\geq 0}$. The Cauchy problem for the ODE $x'(t) = f(t, x)$ has been studied by H. Komatsu (1980) and P. Djakov and B. Mityagin (2002) under some assumptions on $\mathbb{M}$. Finally, V. Thilliez (2010) has obtained results for algebraic equations, both in the quasianalytic ultradifferentiable case and in the ultraholomorphic case for strongly regular sequences.

Setting of the problem

The motivation comes from the paper
A. Lastra, S. Malek, Multi-level Gevrey solutions of singularly perturbed linear partial differential equations, submitted, available at http://arxiv.org/abs/1407.2008

We study a family of linear PDEs of the form

$$(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a)\partial_z^S X(t,z,\epsilon)=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon), \tag{1}$$

for given initial data

$$\partial_z^j X(t,0,\epsilon)=\phi_j(t,\epsilon),\quad 0\le j\le S-1. \tag{2}$$

Here, $a\in\mathbb{C}^\star:=\mathbb{C}\setminus\{0\}$;
$\mathcal{S}$ is a finite subset of $\mathbb{N}_0^3$, where $\mathbb{N}_0=\{0,1,2,\ldots\}$;
$S>\kappa_1$ for every $(s,\kappa_0,\kappa_1)\in\mathcal{S}$,
and the initial data are holomorphic functions defined in a product of two finite sectors, $\mathcal{T}$ for the variable t and $\mathcal{E}$ for the perturbation parameter ϵ.

Setting of the problem (continued)

The coefficients $b_{s\kappa_0\kappa_1}$ belong to $\mathcal{O}(D\times\mathcal{E})$, where D is a neighborhood of the origin in z.

Moreover, if one writes

$$b_{s\kappa_0\kappa_1}(z,\epsilon)=\sum_{\beta\geq 0} b_{s\kappa_0\kappa_1\beta}(\epsilon)\frac{z^\beta}{\beta!},$$

for every $(s,\kappa_0,\kappa_1)\in\mathcal{S}$, the function $b_{s\kappa_0\kappa_1\beta}(\epsilon)$ will be the $\mathbb{M}-$sum of a formal power series $\hat{b}_{s\kappa_0\kappa_1\beta}(\epsilon)$ in a direction.

Overview of the results

Several asymptotic phenomena will appear:

- On the one hand, the appearance of an irregular singularity $t^{k+1}\partial_t$ perturbed by a power of ϵ causes a Gevrey-like asymptotics with respect to the perturbation parameter. Several forbidden directions with respect to summability of the formal solution would appear in this respect.
- On the other hand, the nature of the coefficients $b_{s\kappa_0\kappa_1\beta}$ induces an asymptotic behavior for the solution related to the sequence $\mathbb{M}$.
- Under some assumptions, we may speak about multisummability of the existing formal solution.

Strongly regular sequences (following V. Thilliez)

A sequence $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ of positive real numbers, with $M_0 = 1$, is said to be strongly regular if it is:

- logarithmically convex: $M_n^2 \le M_{n-1}M_{n+1}, \quad n \ge 1.$
- of moderate growth: there exists a constant $A > 0$ such that

$$M_{n+p} \le A^{n+p} M_n M_p, \quad n, p \in \mathbb{N}_0.$$

- strongly non-quasianalytic: there exists $B > 0$ such that

$$\sum_{k\ge n} \frac{M_k}{(k+1)M_{k+1}} \le B\frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0.$$

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Examples:

- $\mathbb{M}_\alpha = (n!^\alpha)_{n\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$.
- $\mathbb{M}_{\alpha,\beta} = \left(n!^\alpha \prod_{m=0}^n \log^\beta(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

Asymptotics and quasianalyticity

We say a holomorphic function f in a sectorial region G admits the series $\hat{f}=\sum_{n=0}^{\infty}a_nz^n$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f\sim_{\mathbb{M}}\hat{f}$, if for every $T\ll G$ there exist $C_T,B_T>0$ such that for every $z\in T$ and every $n\in\mathbb{N}_0$, we have

$$\Big|f(z)-\sum_{k=0}^{n-1}a_kz^k\Big|\le C_TB_T^nM_n|z|^n. \qquad [f\in\tilde{\mathcal{A}}_{\mathbb{M}}(G)]$$

$f\in\tilde{\mathcal{A}}_{\mathbb{M}}(G)\Leftrightarrow$ for every $T\ll G$ there exist $C_T,B_T>0$ such that $|f^{(n)}(z)|\le C_TB_T^nn!M_n$, $n\in\mathbb{N}_0,z\in T$.

Asymptotics and quasianalyticity

We say a holomorphic function f in a sectorial region G admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f \sim_{\mathbb{M}} \hat{f}$, if for every $T \ll G$ there exist $C_T, B_T > 0$ such that for every $z \in T$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \le C_T B_T^n M_n |z|^n. \qquad [f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)]$$

$f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G) \Leftrightarrow$ for every $T \ll G$ there exist $C_T, B_T > 0$ such that $|f^{(n)}(z)| \le C_T B_T^n n! M_n$, $n \in \mathbb{N}_0, z \in T$.

$f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ is said to be flat if $f \sim_{\mathbb{M}} \hat{0}$, the null series.

This amounts to: For every bounded proper subsector T of G there exist $c_1, c_2 > 0$ with

$$|f(z)| \le c_1 h_{\mathbb{M}}(c_2|z|), \qquad z \in T,$$

where $h_{\mathbb{M}}(t) = \inf_{n \ge 0} M_n t^n$, $t > 0$; $h_{\mathbb{M}}(0) = 0$.

Optimal opening for quasianalyticity

The class $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ is said to be quasianalytic if it does not contain nontrivial flat functions.
G_γ will denote any sectorial region of opening $\pi\gamma$.

B. I. Korenbljum, Conditions of nontriviality of certain classes of functions analytic in a sector, and problems of quasianalyticity, Soviet Math. Dokl. 7 (1966), 232–236.
J. S., Flat functions in Carleman ultraholomorphic classes via proximate orders, J. Math. Anal. Appl. 415 (2014), 623–643.

Proposition (J.S. (2014))

For $\mathbb{M}$ logarithmically convex, we have

$$\omega(\mathbb{M}) := \inf\{\gamma > 0 : \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma) \text{ is quasianalytic}\} = \liminf_{n\to\infty} \frac{\log(M_{n+1}/M_n)}{\log(n)}.$$

If $\mathbb{M}$ is strongly regular, then $\omega(\mathbb{M}) \in (0,\infty)$.

Watson's Lemma whenever $d(r)$ is a proximate order

Put $d(r) = \dfrac{\log\big(-\log\big(h_{\mathbb{M}}(1/r)\big)\big)}{\log(r)}$, $r > 0$ large enough.

Theorem (generalized Watson's lemma, partial version, J.S. (2014))

Suppose $d(r)$ is a proximate order. Then, $\tilde{A}_{\mathbb{M}}(G)$ is quasianalytic if, and only if, the opening of G is greater than $\pi\omega(\mathbb{M})$.

Proposition (J. Jiménez-Garrido, J.S.(2015))

Let $\mathbb{M}$ be a strongly regular sequence, the following statements are equivalent:

(i) $d(r)$ *is a proximate order.*

(ii) $\displaystyle\lim_{n\to\infty} \frac{\log(M_{n+1}/M_n)}{\log(n)} = \omega(\mathbb{M})$.

All the examples studied satisfy this condition.

Kernels and transforms for $\mathbb{M}$-summability

A. Lastra, S. Malek, J. S., Summability in general Carleman ultraholomorphic classes, J. Math. Anal. Appl. 430 (2015), 1175–1206.

Suppose $\mathbb{M}$ is a strongly regular sequence such that $d(r)$ is a proximate order. Then:

- One can define kernels for $\mathbb{M}$-summability, e and E, with properties similar to those introduced by W. Balser for his kernels of order k (in the Gevrey case).
- One may construct Laplace- and Borel-like transforms, both formal and analytic, which behave as the classical ones in $k-$summability, so that one can introduce a satisfactory theory of $\mathbb{M}-$summability.

Definition of $\mathbb{M}$-summability in a direction and main result

Definition (A. Lastra, S. Malek, J.S. (2015))

Let $d \in \mathbb{R}$. We say $\hat{f} = \sum_{n\geq 0} \dfrac{f_n}{n!} z^n$ is $\mathbb{M}$-summable in direction d if there exist a sectorial region $G = G(d,\gamma)$, with $\gamma > \omega(\mathbb{M})$, and $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ such that $f \sim_{\mathbb{M}} \hat{f}$.

In this case, f is unique and will be called the $\mathbb{M}$-sum of $\hat{f}$ in direction d.

Definition of $\mathbb{M}$-summability in a direction and main result

Definition (A. Lastra, S. Malek, J.S. (2015))

Let $d \in \mathbb{R}$. We say $\hat{f} = \sum_{n\geq 0} \frac{f_n}{n!} z^n$ is $\mathbb{M}$-summable in direction d if there exist a sectorial region $G = G(d,\gamma)$, with $\gamma > \omega(\mathbb{M})$, and $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ such that $f \sim_{\mathbb{M}} \hat{f}$.

In this case, f is unique and will be called the $\mathbb{M}$-sum of $\hat{f}$ in direction d.

Theorem (A. Lastra, S. Malek, J.S. (2015))

Given $\mathbb{M}$, d and $\hat{f} = \sum_{n\geq 0} \frac{f_n}{n!} z^n$, the following are equivalent:

- *$\hat{f}$ is $\mathbb{M}$-summable in direction d.*
- *For every (some) kernel e of $\mathbb{M}-$summability, its formal Borel transform converges and it admits analytic continuation in an unbounded sector bisected by d, where it has a suitable growth (in terms of $\mathbb{M}$).*

In case any of the previous holds, the $\mathbb{M}$-sum of $\hat{f}$ in direction d is the Laplace transform of g.

First transformation, second problem

Let $\epsilon \in \mathcal{E}$. By making the change of variable

$$T := \epsilon^r t, \qquad Y(T,z,\epsilon) := X(t,z,\epsilon),$$

we transform our initial problem into an equivalent one for Y, namely

$$\begin{aligned} &((T^{k+1}\partial_T)^{s_1} + a)\partial_z^S Y(T,z,\epsilon) \\ &\qquad = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)} T^s (\partial_T^{\kappa_0}\partial_z^{\kappa_1} Y)(T,z,\epsilon), \end{aligned} \tag{3}$$

with suitable initial conditions

$$(\partial_z^j Y)(T,0,\epsilon) = Y_j(T,\epsilon), \quad 0 \le j \le S-1. \tag{4}$$

Let $k \ge 1$ be an integer, and $m_k(n) = \Gamma\left(\frac{n}{k}\right)$, $n \ge 1$. We consider m_k-summability (in the sense of W. Balser) of formal power series with coefficients in a complex Banach space, along with the (formal) Borel transform, $\mathcal{B}_{m_k}$, and Laplace transform in a direction, $\mathcal{L}^d_{m_k}$.

Second transformation, third problem

Applying $\mathcal{B}_{m_k}$ to (3), (4), we obtain for $W(\tau,z,\epsilon) := \mathcal{B}_{m_k}(Y(T,z,\epsilon))$ and every $\epsilon \in \mathcal{E}$ the Cauchy problem

$$((k\tau^k)^{s_1} + a)\partial_z^S W(\tau,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)} \tag{5}$$

$$\times \left[\frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)} \int_0^{\tau^k} (\tau^k - \sigma)^{\frac{\delta_{\kappa_0}}{k}-1} (k\sigma)^{\kappa_0} \partial_z^{\kappa_1} W(\sigma^{1/k}, z, \epsilon) \frac{d\sigma}{\sigma} \right.$$

$$+ \sum_{1\le p\le \kappa_0-1} A_{\kappa_0,p} \frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}$$

$$\left. \times \int_0^{\tau^k} (\tau^k-\sigma)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1} (k\sigma)^p \partial_z^{\kappa_1} W(\sigma^{1/k}, z, \epsilon) \frac{d\sigma}{\sigma} \right],$$

for initial data

$$(\partial_z^j) W(\tau, 0, \epsilon) = W_j(\tau,\epsilon), \quad 0 \le j \le S-1. \tag{6}$$

Initial conditions for the third problem

Let $\epsilon \in \mathcal{E}$ and $r \in \mathbb{Q}$, $r > 0$.
For every $\beta \geq 0$, we consider the Banach space $F_{\beta,\epsilon,\Omega}$, consisting of the holomorphic functions defined in $\Omega := S_d \cup D(0, \rho_0)$, $\tau \mapsto h(\tau, \epsilon)$ such that

$$\|h(\tau,\epsilon)\|_{\beta,\epsilon,\Omega} := \sup_{\tau\in\Omega} \left\{ \frac{1 + \left|\frac{\tau}{\epsilon^r}\right|^{2k}}{\left|\frac{\tau}{\epsilon^r}\right|} \exp\left(-\sigma r_b(\beta) \left|\frac{\tau}{\epsilon^r}\right|^k \right) |h(\tau,\epsilon)| \right\} < \infty,$$

where $r_b(\beta) = \sum_{n=0}^{\beta}(n+1)^{-b}$.

We assume that the initial data $W_j(\tau,\epsilon) \in F_{j,\epsilon,\Omega}$, and

$$\sup_{\epsilon\in\mathcal{E}} \|W_j(\tau,\epsilon)\|_{j,\epsilon,\Omega} < \infty, \quad 0 \leq j \leq S-1.$$

Assumptions

Assumption (A): Let $a \in \mathbb{C}$ with $a \neq 0$, and let s_1 be a positive integer. We assume that $ks_1 \arg(\tau) \neq \pi(2j+1) + \arg(a)$ for $j = 0, \dots, ks_1 - 1$ and every $\tau \in \overline{S_d} \setminus \{0\}$. In addition to this, we take ρ_0 under the condition $\rho_0 < \frac{|a|^{1/ks_1}}{2k^{1/k}}$.

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Let S, r_1 be positive integers, and put $r := \dfrac{r_1}{s_1 k}$.
Assumption (B): For every $(s, \kappa_0, \kappa_1) \in \mathcal{S}$ there exists a nonnegative integer $\delta_{\kappa_0} \geq k$ such that

$$s = \kappa_0(k+1) + \delta_{\kappa_0}.$$

Solution to the third problem

Proposition

Under Assumptions (A) and (B), there exists a formal power series solution of (5), (6),

$$W(\tau,z,\epsilon)=\sum_{\beta\geq 0} W_{\beta}(\tau,\epsilon)\frac{z^{\beta}}{\beta!},$$

where $W_{\beta}(\tau,\epsilon)\in F_{\beta,\epsilon,\Omega}$ *for every* $\epsilon\in\mathcal{E}$, $\beta\geq 0$. *Moreover, there exist* $Z_0,Z_1>0$ *such that*

$$\|W_{\beta}(\tau,\epsilon)\|_{\beta,\epsilon,\Omega}\leq Z_1 Z_0^{\beta}\beta!$$

for every $\beta\geq 0$ *and every* $\epsilon\in\mathcal{E}$.

Solution to the second problem

Theorem

Let $\epsilon \in \mathcal{E}$. Problem (3), (4) admits a holomorphic solution

$$(T, z) \mapsto Y(T, z, \epsilon) := \sum_{\beta \geq 0} \mathcal{L}^{d}_{m_k}(W_\beta(\tau, \epsilon))(T) \frac{z^\beta}{\beta!}$$

defined in

$$S \times D(0, \frac{1}{Z_0}),$$

for some bounded sector S bisected by d with opening larger than π/k, and Z_0 as before.

Good covering and associated family

Definition

We say a family $(\mathcal{E}_i)_{0\leq i\leq \nu-1}$ of open sectors with vertex at the origin and finite radius $r_{\mathcal{E}}>0$ provides a good covering in $\mathbb{C}^\star$ if $\mathcal{E}_i\cap\mathcal{E}_{i+1}\neq\emptyset$ for $0\leq i\leq \nu-1$ (where $\mathcal{E}_\nu:=\mathcal{E}_0$) and $\cup_{0\leq i\leq\nu-1}\mathcal{E}_i=\mathcal{U}\setminus\{0\}$ for some neighborhood of the origin $\mathcal{U}$.

Let $\mathcal{T}$ be a bounded sector, and for $0\leq i\leq\nu-1$ let $\tilde{S}_{d_i}$ be a bounded sector bisected by direction d_i, with opening larger than π/k, and such that:

- One has $d_i\neq\frac{\pi(2j+1)+\arg(a)}{ks_1}$ for all $j=0,\dots,ks_1-1$.
- For every $0\leq i\leq\nu-1$, $t\in\mathcal{T}$ and $\epsilon\in\mathcal{E}_i$ one has $\epsilon^r t\in\tilde{S}_{d_i}$.

We say that $((\tilde{S}_{d_i})_{0\leq i\leq\nu-1},\mathcal{T})$ is associated with the good covering $(\mathcal{E}_i)_{0\leq i\leq\nu-1}$.

Solution to the first problem, I

Assume that $\omega(\mathbb{M}) < 1/(rk)$.

Let $(\mathcal{E}_i)_{0\leq i\leq \nu-1}$ be a good covering such that the opening of every $\mathcal{E}_i$ is larger than $\pi\omega(\mathbb{M})$; and $((\tilde{S}_{d_i})_{0\leq i\leq \nu-1}, \mathcal{T})$ be associated with it. We put $\Omega_i := S_{d_i} \cup D(0,\rho_0)$, for some unbounded sector S_{d_i} around d_i with small opening.

For every $(s,\kappa_0,\kappa_1) \in \mathcal{S}$, let

$$\hat{b}_{s\kappa_0\kappa_1}(z,\epsilon) = \sum_{\beta\geq 0} \hat{b}_{s\kappa_0\kappa_1\beta}(\epsilon)\frac{z^\beta}{\beta!}$$

be $\mathbb{M}-$summable on $\mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$, with sum $b^{(i)}_{s\kappa_0\kappa_1}(z,\epsilon) = \sum_{\beta\geq 0} b^{(i)}_{s\kappa_0\kappa_1\beta}(\epsilon)\frac{z^\beta}{\beta!}$ (holomorphic in $\mathcal{E}_i$ with values in $\mathcal{O}(D)$).

So, for all $\beta \geq 0$ and some $c_1, c_2, c_3 > 0$, we have

$$|b^{(i+1)}_{s\kappa_0\kappa_1\beta}(\epsilon) - b^{(i)}_{s\kappa_0\kappa_1\beta}(\epsilon)| \leq c_1 c_2^\beta \beta! h_{\mathbb{M}}(c_3|\epsilon|), \quad \epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}.$$

Solution to the first problem, II

For every $0 \leq i \leq \nu - 1$, we study the Cauchy problem

$$(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1} + a)\partial_z^S X_i(t,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b^{(i)}_{s\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X_i)(t,z,\epsilon) \tag{1_i}$$

under the initial conditions

$$(\partial_z^j(X_i))(t,0,\epsilon) = \phi_{i,j}(t,\epsilon), \quad 0 \leq j \leq S-1, \tag{2_i}$$

where the functions $\phi_{i,j}$ are constructed as m_k-Laplace transforms of holomorphic functions $W_{i,j} : \Omega_i \times \mathcal{E}_i \to \mathbb{C}$ such that:

- For every $\epsilon \in \mathcal{E}_i$, $W_{i,j}(\tau,\epsilon) \in F_{j,\epsilon,\Omega_i}$, and $\sup_{\epsilon\in\mathcal{E}_i} \|W_{i,j}(\tau,\epsilon)\|_{j,\epsilon,\Omega_i} < \infty$.
- There exists $K > 0$ such that

$$\sup_{\epsilon\in\mathcal{E}_i\cap\mathcal{E}_{i+1}} \frac{\|W_{i+1,j}(\tau,\epsilon) - W_{i,j}(\tau,\epsilon)\|_{j,\epsilon,\Omega_i\cap\Omega_{i+1}}}{h_{\mathbb{M}}(K|\epsilon|)} < \infty.$$

Solution to the first problem, III

Theorem

Under assumptions (A) and (B), the problem $(1_i), (2_i)$ *has a holomorphic and bounded solution* $X_i(t, z, \epsilon)$ *on* $(\mathcal{T} \cap D(0, h')) \times D(0, R_0) \times \mathcal{E}_i$*, for every* $0 \leq i \leq \nu - 1$*, for some* $R_0, h' > 0$*.*

Indeed,

$$X_i(t, z, \epsilon) := Y_i(\epsilon^r t, z, \epsilon) = \sum_{\beta \geq 0} k \int_{L_{\gamma_i}} W_{i,\beta}(u, \epsilon) e^{-\left(\frac{u}{t\epsilon^r}\right)^k} \frac{du}{u} \frac{z^\beta}{\beta!},$$

Solution to the first problem, III

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Proposition

Let $0 \leq i \leq \nu - 1$*. There exist* $\tilde{c}_0, \tilde{c}_1, \tilde{K}_2 > 0$ *such that for every* $\beta \geq 0$ *and* $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ *one has*

$$\|W_{i+1,\beta}(\tau,\epsilon) - W_{i,\beta}(\tau,\epsilon)\|_{\beta,\epsilon,\Omega_i \cap \Omega_{i+1}} \leq \tilde{c}_1 \tilde{c}_0^\beta \beta! h_{\mathbb{M}} \left(\tilde{K}_2 |\epsilon|\right).$$

Difference of neighboring solutions, case I

Theorem

Let $0 \leq i \leq \nu - 1$. Assume there are no singular directions $\frac{\pi(2j+1)+\arg(a)}{ks_1}$ for $j = 0, \dots, ks_1 - 1$ in between γ_i and γ_{i+1}. Then, there exist $\tilde{K}_1, \tilde{K}_2 > 0$ such that

$$|X_{i+1}(t, z, \epsilon) - X_i(t, z, \epsilon)| \leq \tilde{K}_1 h_{\mathbb{M}} \left(\tilde{K}_2 |\epsilon| \right),$$

for every $t \in \mathcal{T} \cap D(0, h')$, $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ and all $z \in D(0, \tilde{R}_0)$, for some $\tilde{R}_0 > 0$.

Difference of neighboring solutions, case II

Theorem

Let $0 \leq i \leq \nu - 1$. Assume now that there exists $j \in \{0, \dots, ks_1 - 1\}$ such that the singular direction $\frac{\pi(2j+1)+\arg(a)}{ks_1}$ lies in between γ_i and γ_{i+1}. Then, there exist $\tilde{K}_3, \tilde{K}_4 > 0$ such that

$$|X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \leq \tilde{K}_3 \exp\left(-\frac{\tilde{K}_4}{|\epsilon|^{rk}}\right),$$

for every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, $t \in \mathcal{T} \cap D(0,h')$, and all $z \in D(0,R_0')$, for some $R_0' > 0$.

There exist $K, K' > 0$ such that

$$h_{\mathbb{M}}(K|\epsilon|) \leq e^{-K'/|\epsilon|^{rk}},$$

Ramis-Sibuya revisited

Lemma

Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence, and let $(\mathcal{E}_i)_{0\leq i\leq\nu-1}$ be a good covering in $\mathbb{C}^\star$. Assume there exist $f_1,\ldots,f_\nu$ such that:

(i) f_ℓ is holomorphic in $\mathcal{E}_{\ell-1}\cap\mathcal{E}_\ell$ for every $\ell = 1,\ldots,\nu$ (where $\mathcal{E}_\nu := \mathcal{E}_0$).

(ii) There exist $C_1, C_2 > 0$ such that

$$|f_\ell(\epsilon)| \leq C_1 h_{\mathbb{M}}(C_2|\epsilon|), \quad \epsilon\in\mathcal{E}_{\ell-1}\cap\mathcal{E}_\ell,\ \ell = 1,\ldots,\nu.$$

Then, there exist $\psi_0,\ldots,\psi_{\nu-1}$ and a formal power series $\hat{\psi} = \sum_{p\geq 0} a_p\epsilon^p \in \mathbb{C}[[\epsilon]]$ such that

(i) ψ_ℓ admits $\hat{\psi}$ as its $\mathbb{M}$-asymptotic expansion in $\mathcal{E}_\ell$ for all $\ell = 0,\ldots,\nu-1$.

(ii) $f_\ell(\epsilon) = \psi_\ell(\epsilon) - \psi_{\ell-1}(\epsilon)$ for $\epsilon\in\mathcal{E}_{\ell-1}\cap\mathcal{E}_\ell$, $0\leq\ell\leq\nu-1$.

Given $K_1 > 0$ there exist $K_2, K_3 > 0$ such that for every $p\in\mathbb{N}$ one has

$$\int_0^\infty t^{p-1} h_{\mathbb{M}}(K_1/t)\,dt \leq K_2 K_3^p M_p.$$

A definition of multisummability

Definition

Let $(\mathbb{E}, ||.||_{\mathbb{E}})$ be a complex Banach space, let $\kappa > 0$ and let $\mathbb{M} = (M_p)_{p\geq 0}$ be a strongly regular sequence such that $\omega(\mathbb{M}) < 1/\kappa$.
Let $\mathcal{E}$ be a bounded open sector centered at 0 with opening greater than $\pi\omega(\mathbb{M})$, and let $\mathcal{F}$ be a bounded open sector centered at 0 with opening greater than $\frac{\pi}{\kappa}$, such that $\mathcal{E} \subset \mathcal{F}$.
A formal power series $\hat{f}(\epsilon) = \sum_{n\geq 0} a_n \epsilon^n \in \mathbb{E}[[\epsilon]]$ is said to be $(\mathbb{M}, \kappa)-$summable on $\mathcal{E}$ if there exist a formal series $\hat{f}_1(\epsilon) \in \mathbb{E}[[\epsilon]]$ which is $\mathbb{M}-$summable on $\mathcal{E}$ with $\mathbb{M}-$sum $f_1 : \mathcal{E} \to \mathbb{E}$ and a formal series $\hat{f}_2(\epsilon) \in \mathbb{E}[[\epsilon]]$ which is $\kappa-$summable on $\mathcal{F}$ with $\kappa-$sum $f_2 : \mathcal{F} \to \mathbb{E}$ such that $\hat{f} = \hat{f}_1 + \hat{f}_2$.
Furthermore, the holomorphic function $f(\epsilon) = f_1(\epsilon) + f_2(\epsilon)$ defined on $\mathcal{E}$ is called the $(\mathbb{M}, \kappa)-$sum of $\hat{f}$ on $\mathcal{E}$.

A version of Ramis-Sibuya theorem, I

Theorem

Let $\mathbb{E}$ be a complex Banach space, $(\mathcal{E}_i)_{0\leq i\leq \nu-1}$ a good covering in $\mathbb{C}^{\star}$ and $\mathbb{M}$ a strongly regular sequence. Consider $G_i : \mathcal{E}_i \to \mathbb{E}$, a holomorphic function for all $0 \leq i \leq \nu - 1$, and put $\Delta_i(\epsilon) := G_{i+1}(\epsilon) - G_i(\epsilon)$ for every $\epsilon \in Z_i := \mathcal{E}_i \cap \mathcal{E}_{i+1}$. We assume:

(1) *The functions $G_i(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_i$ tends to 0.*

(2) *There is a nontrivial partition I_1, I_2 of $\{0, \dots, \nu - 1\}$ such that for some $\alpha > 0$ and every $i \in I_1$ there exist $K_1, M_1 > 0$ such that*

$$\|\Delta_i(\epsilon)\| \leq K_1 e^{-\frac{M_1}{|\epsilon|^{\alpha}}}, \quad \epsilon \in Z_i,$$

and for every $i \in I_2$ there exist $K_2, M_2 > 0$ such that

$$\|\Delta_i(\epsilon)\| \leq K_2 h_{\mathbb{M}}(M_2|\epsilon|), \quad \epsilon \in Z_i.$$

A version of Ramis-Sibuya theorem, II

Theorem

Then, there exists $a(\epsilon) \in \mathbb{E}\{\epsilon\}$ *and* $\hat{G}^1(\epsilon), \hat{G}^2(\epsilon) \in \mathbb{E}[[\epsilon]]$ *such that*

$$G_i(\epsilon) = a(\epsilon) + G_i^1(\epsilon) + G_i^2(\epsilon),$$

where $G_i^1(\epsilon)$ *is holomorphic in* $\mathcal{E}_i$ *and has* $\hat{G}^1(\epsilon)$ *as its* $1/\alpha$*-Gevrey asymptotic expansion on* $\mathcal{E}_i$ *for every* $i \in I_1$*, while* $G_i^2(\epsilon)$ *is holomorphic on* $\mathcal{E}_i$ *and has* $\hat{G}^2(\epsilon)$ *as its* $\mathbb{M}-$*asymptotic expansion on* $\mathcal{E}_i$ *for* $i \in I_2$*.*
Assume moreover that $\omega(\mathbb{M}) < 1/\alpha$*, and some integer* $i_0 \in I_2$ *is such that* $I_{\delta_1,i_0,\delta_2} = \{i_0 - \delta_1, \dots, i_0, \dots, i_0 + \delta_2\} \subset I_2$ *for some integers* $\delta_1, \delta_2 \geq 0$ *and with the property that*

$$\mathcal{E}_{i_0} \subset S \subset \bigcup_{h \in I_{\delta_1,i_0,\delta_2}} \mathcal{E}_h$$

where $\mathcal{E}_{i_0}$ *has opening larger than* $\pi\omega(\mathbb{M})$ *and* S *has opening larger than* π/α*.*
Then, the formal series $\hat{G}(\epsilon)$ *is* $(\mathbb{M}, \alpha)-$*summable on* $\mathcal{E}_{i_0}$*, with* $(\mathbb{M}, \alpha)-$*sum* $G_{i_0}(\epsilon)$*.*

Final statement

$\mathbb{E}$ is the Banach space (with the supremum norm) of bounded holomorphic functions on $(\mathcal{T} \cap D(0,h')) \times D(0,R_0)$, where h' and R_0 are suitable constants.

Theorem

There exists a formal solution $\hat{X}(t,z,\epsilon) = \sum_{\beta \geq 0} H_\beta(t,z) \frac{\epsilon^\beta}{\beta!} \in \mathbb{E}[[\epsilon]]$, *of*

$$(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1} + a)\partial_z^S \hat{X}(t,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} \hat{b}_{s\kappa_0\kappa_1}(z,\epsilon) t^s (\partial_t^{\kappa_0}\partial_z^{\kappa_1}\hat{X})(t,z,\epsilon).$$

Moreover, we can write $\hat{X}(t,z,\epsilon) = a(t,z,\epsilon) + \hat{X}^1(t,z,\epsilon) + \hat{X}^2(t,z,\epsilon)$, *where* $a(t,z,\epsilon) \in \mathbb{E}\{\epsilon\}$, $\hat{X}^1(t,z,\epsilon)$ *and* $\hat{X}^2(t,z,\epsilon) \in \mathbb{E}[[\epsilon]]$. *For every* $0 \leq i \leq \nu - 1$, *the analytic solution* $\epsilon \mapsto X_i(t,z,\epsilon)$ *may be given as*

$$X_i(t,z,\epsilon) = a(t,z,\epsilon) + X_i^1(t,z,\epsilon) + X_i^2(t,z,\epsilon),$$

where $\epsilon \mapsto X_i^1(t,z,\epsilon)$ *admits* $\hat{X}^1(t,z,\epsilon)$ *as* $1/(rk)-$*Gevrey asymp. exp. in* $\mathcal{E}_i$, *and* $\epsilon \mapsto X_i^2(t,z,\epsilon)$ *admits* $\hat{X}^2(t,z,\epsilon)$ *as* $\mathbb{M}-$*asymp. exp. in* $\mathcal{E}_i$.

Final statement (continued)

Theorem

Moreover, assume there exist $0 \leq i_0 \leq \nu - 1$ *and two integers* $\delta_1, \delta_2 \geq 0$ *such that for all* $h \in I_{\delta_1, i_0, \delta_2} = \{i_0 - \delta_1, \dots, i_0, \dots, i_0 + \delta_2\}$, *there are no singular directions* $\frac{\pi(2j+1)+\arg(a)}{ks_1}$, *for any* $0 \leq j \leq ks_1 - 1$, *in between* γ_h *and* γ_{h+1}, *and such that*

$$\mathcal{E}_{i_0} \subset S \subset \bigcup_{h \in I_{\delta_1, i_0, \delta_2}} \mathcal{E}_h,$$

where S *is a sector with opening larger than* $\frac{\pi}{rk}$. *Then, the formal series* $\hat{X}(t,z,\epsilon)$ *is* $(\mathbb{M}, rk)$*-summable on* $\mathcal{E}_{i_0}$ *and its* $(\mathbb{M}, rk)$*-sum is given by* $X_{i_0}(t,z,\epsilon)$.

THANK YOU VERY MUCH FOR YOUR ATTENTION

DZIEKUJE BARDZO ZA UWAGE