Higher order Painlevé systems, rigid systems and hypergeometric functions

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1 Painlevé Equations

Before stating our aim, we shall review how the Painlevé equation are discovered.
Let us consider

Problem 1.1. *Define a new transcendental function as a solution of a differential equation in the complex domain.*

Then the solution should be controlled by the differential equation.
But it is hard to know where the solution appears if the equation is non-linear.

Example 1.2. *The following differential equation has a movable branch point:*

$$ny^{n-1}\frac{dy}{dt} = 1 \quad (n \in \mathbb{N}).$$

In fact, it has a solution $y = (t-c)^{1/n}$.

Thus we require that the differential equation has no movable branch point (the Painlevé property).

Painlevé and Gambier tried to classify all of 2nd order ordinary differential equations with the Painlevé property.

Fact 1.3 ([Fuchs-Poincaré 19c]). *All differential equations of the form*

$$R(t, y, y') = 0 \quad (t \in \mathbb{P}^1(\mathbb{C})),$$

with the Painlevé property are reduced to the 3 types of equations:

① *Solvable by quadratures*

② $y' = a(t)y^2 + b(t)y + c(t)$

③ $y' = 4y^3 - g_2 y - g_3 \ (g_2, g_3 \in \mathbb{C})$

Fact 1.4 ([Painlevé-Gambier 1910]). *All differential equations of the form*

$$y'' = R(t, y, y') \quad (t \in \mathbb{P}^1(\mathbb{C})),$$

with the Painlevé property are reduced to the 4 types of equations:

① *Solvable by quadratures*

② *Linear differential equations*

③ $y'' = 6y^2 - g_2 \ (g_2 \in \mathbb{C})$

④ *Painlevé equations* $P_{\mathrm{I}}, \ldots, P_{\mathrm{VI}}$

Fact 1.5 ([Okamoto 1980]). *The Painlevé equations are described in the form of polynomial Hamiltonian systems*

$$\frac{dq}{dt} = \frac{\partial H_{\mathrm{J}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\mathrm{J}}}{\partial q} \quad (\mathrm{J} = \mathrm{I}, \ldots, \mathrm{VI}),$$

with

$$H_{\mathrm{I}} = \frac{1}{2}p^2 - 2q^3 - tq, \quad H_{\mathrm{II}} = \frac{1}{2}p(p - 2q^2 - t) - \alpha q,$$
$$tH_{\mathrm{III}} = qp(qp - tq - \alpha) + \beta tq + tp, \quad H_{\mathrm{IV}} = qp(p - q - t) - \alpha q - \beta p,$$
$$tH_{\mathrm{V}} = q(q-1)p(p+t) + \alpha tq + \beta p - \gamma qp,$$
$$t(t-1)H_{\mathrm{VI}} = q(q-1)(q-t)p\left(p - \frac{\alpha}{q-t} - \frac{\beta}{q-1} - \frac{\gamma}{q}\right) + \delta q,$$

where $\alpha, \beta, \gamma, \delta$ are complex constants.

As is seen later, the obtained equations in this talk can be regarded as extensions of these Hamiltonian systems (especially H_{VI}).

We next want to consider

Problem 1.6. *Classify all higher order differential equations with the Painlevé properties.*

This problem hasn't been solved yet (quite difficult!), but we have obtaind the 3 types of generalizations of the Painlevé equations.

① Holonomic deformations of ordinary linear differential equations
[Garnier 1912] [Kimura 1989] [Sakai 2010] etc.

② Similarity reductions of infinite dimensional integrable hierarchies
[Adler 1994] [Gordoa-Joshi-Pickering 2001] [Fuji-S 2007] [Tsuda 2010] etc.

③ Affine Weyl group symmetries
[Noumi-Yamada 1998] [Sasano 2005] etc.

In this talk we change the point of view slightly and propose a generalization from a viewpoint of hypergeometric functions.

2 From Gauss to Painlevé

In order to state our motivation more precisely, we shall review the relationship between the Gauss hypergeometric function ${}_2F_1$ and P_{VI}.
${}_2F_1$ satisfies the hypergeometric differential equation (HGDE)

$$\{(x\partial_x + \alpha)(x\partial_x + \beta) - \partial_x(x\partial_x + \gamma - 1)\}\, y = 0,$$

which can be rewritten into a (Schlesinger type) Fuchsian system

$$\partial_x \mathbf{y} = \left(\frac{1}{x-1}\begin{bmatrix} -\beta+\gamma-1 & -\alpha \\ -\beta+\gamma-1 & -\alpha \end{bmatrix} + \frac{1}{x}\begin{bmatrix} -\gamma+1 & \alpha \\ 0 & 0 \end{bmatrix}\right)\mathbf{y}.$$

In the following we regard this Fuchsian system as HGDE.
Its Riemann scheme is given by

$$\left\{\begin{matrix} x=1 & x=0 & x=\infty \\ -\alpha-\beta+\gamma-1 & -\gamma+1 & \alpha \\ 0 & 0 & \beta \end{matrix}\right\}.$$

Notice that HGDE is determined uniquely by this Riemann scheme.

By adding 1 regular singularity $x = t$, we consider a Fuchsian system with a Riemann scheme

$$\left\{ \begin{matrix} x = t & x = 1 & x = 0 & x = \infty \\ \eta & -\alpha - \beta + \gamma - 1 & -\gamma + 1 & \alpha \\ 0 & 0 & 0 & \beta \end{matrix} \right\}.$$

This system contains 2 parameters which aren't determined by this Riemann scheme (accessory parameters).

Denoting the accessory parameters by q and p, we can take residue matrices as

$$\partial_x \mathbf{y} = \left(\frac{A_t}{x - t} + \frac{A_1}{x - 1} + \frac{A_0}{x} \right) \mathbf{y}, \quad A_t = \begin{bmatrix} -(qp + \eta) & p \\ -q(qp + \eta) & qp \end{bmatrix},$$

$$A_1 = \begin{bmatrix} qp + \eta - \beta + \gamma - 1 & -qp - \alpha \\ qp + \eta - \beta + \gamma - 1 & -qp - \alpha \end{bmatrix}, \quad A_0 = \begin{bmatrix} -\gamma + 1 & (q - 1)p + \alpha \\ 0 & 0 \end{bmatrix}. \qquad (L_{\mathrm{VI}})$$

As the solution $\mathbf{y}$ of L_{VI} depends on q, p, t, the isomonodromy deformation becomes important.

Then we arrive at

Fact 2.1 ([Fuchs 1907]). *The isomonodromy deformation of L_{VI} implies P_{VI}.*

Fact 2.2 ([Fuchs 1907]). *$P_{\rm VI}$ includes HGDE as a particular solution.*

An explicit formula of $P_{\rm VI}$ is given by

$$\begin{aligned} t(t-1)q' &= 2pq^3 - \{2(t+1)p - \alpha - \eta\}q^2 \\ &\qquad + \{2tp - (\alpha - \beta + \eta)t - \gamma - \eta + 1\}q - (\beta - \gamma - \eta + 1)t, \\ t(t-1)p' &= -\{3q^2 - 2(t+1)q + t\}p^2 \\ &\qquad - \{2(\alpha + \eta)q - (\alpha - \beta + \eta)t - \gamma - \eta + 1\}p - \alpha\eta. \end{aligned}$$

We now assume that $p = \eta = 0$. Then we obtain a Riccati equaltion

$$t(t-1)q' = \alpha q^2 - \{(\alpha - \beta)t + \gamma - 1\}q - (\beta - \gamma + 1)t,$$

which is transformed to the HGDE (with $x = t$)

$$\partial_t \, \mathbf{y} = \left(\frac{1}{t-1} \begin{bmatrix} -\beta + \gamma - 1 & -\alpha \\ -\beta + \gamma - 1 & -\alpha \end{bmatrix} + \frac{1}{t} \begin{bmatrix} -\gamma + 1 & \alpha \\ 0 & 0 \end{bmatrix} \right) \mathbf{y}.$$

Summarizing the above, we have the following scheme.

$$
\begin{array}{ccc}
\boxed{\text{HGDE}} & \xrightarrow{\text{add 1 regular singularity}} & \boxed{\text{Fuchsian system } L_{\mathrm{VI}}} \\
\text{particular solution } \nwarrow & & \swarrow \text{ isomonodromy deformation} \\
& \boxed{\text{Painlevé equation } P_{\mathrm{VI}}} &
\end{array}
$$

We want to consider higher order generalizations of such a classical scheme.

- ◦ There exist several generalizations of ${}_2F_1$ (Thomae's ${}_{n+1}F_n$, Appell's $F_1, \ldots, F_4$, Lauricella's $F_A, \ldots, F_D$ and more).
- ◦ Let us try to formulate higher order Painlevé type differential equations which admit particular solutions in terms of those hypergeometric functions.
- ◦ For a systematic investigation, we focus on the isomonodromy deformations of Fuchsian systems.

3 Higher Order Painlevé Systems and Rigid Systems

Fortunately we have a valuable classification theory of isomonodromy deformation equations of Fuchsian systems (the Schlesinger systems).

Fact 3.1 ([Oshima 2008]). *Irreducible Fuchsian systems with a fixed number of accessory parameters can be reduced to finite types of systems by using the Katz's two operations, namely addition and middle convolution.*

This fact gives a classification theory of all irreducible Fuchsian systems.

Fact 3.2 ([Haraoka-Filipuk 2007]). *The isomonodromy deformation equation is invariant under the Katz's two operations.*

This fact lifts the Oshima's classification up to the level of the Schlesinger systems. Combining those 2 previous works, we know how many Schlesinger systems exist.

But to give their explicit formulas is another problem.

- ◦ The construction method of canonical variables for the Schlesinger systems was established by Jimbo, Miwa, Mori and Sato in 1980.
- ◦ But in their Hamiltonian systems the number of canonical variables is larger than the one of accessory parameters.
- ◦ Thus we have to decrease it, but this process doesn't go smoothly due to the technical reason.
- ◦ We haven't found yet an algorithm which gives canonical variables for all possible Schlesinger systems.

Thus we next consider

Problem 3.3. *Describe any Schlesinger system in the form of polynomial Hamiltonian system with the same number of canonical variables as accessory parameters (a higher order Painlevé system).*

This problem is solved partially.

In order to introduce the obtained result, we recall a spectral type of Fuchsian system by using an example.

Example 3.4. *The spectral type of a Fuchsian system with a Riemann scheme*

$$\left\{ \begin{matrix} x=t & x=1 & x=0 & x=\infty \\ \theta_1 & \theta_2 & \kappa_1 & \rho_1 \\ 0 & 0 & \kappa_2 & \rho_2 \\ 0 & 0 & 0 & \rho_3 \end{matrix} \right\},$$

is $\{21, 21, 111, 111\}$.

There exists only 1 type of 2nd order Painlevé system [Kostov 2001].
It consists with the Painlevé-Gambier's classification.

Spectral type	Painlevé system	Reference
$11, 11, 11, 11$	P_{VI}	[Fuchs 1907]

Notice that the other Painlevé equations $(P_{\mathrm{I}}, \ldots, P_{\mathrm{V}})$ are obtained from P_{VI} via degeneration limiting procedures.

There exist 4 types of 4th order Painlevé system [Oshima 2008].
Sakai investigated them systematically.

Spectral type	Painlevé system	Reference
$11, 11, 11, 11, 11$	Garnier	[Garnier 1912]
$21, 21, 111, 111$	Fuji-Suzuki-Tsuda	[Sakai 2010] [Tsuda 2010]
$22, 22, 22, 211$	Matrix type	[Sakai 2010]
$31, 22, 22, 1111$	Sasano	[Sakai 2010]

Notice that Kawakami, Nakamura and Sakai are investigating their degeneration scheme.

There exist 12 types of 6th order Painlevé system [Oshima 2008].
Developing Sakai's method, we can calculate them with large effort and patience.

Spectral type	Painlevé system	Reference
$11, 11, 11, 11, 11, 11^{*1}$	Garnier	[Garnier 1912]
$21, 21, 21, 21, 111$		[S 2012]
$31, 31, 22, 22, 22$		[S 2012]
$21, 111, 111, 111$		[S 2012]
$22, 22, 211, 211$		[S 2012]
$22, 22, 22, 1111$		[S 2012]
$31, 22, 211, 1111$		[S 2012]
$31, 31, 1111, 1111^{*1}$	Fuji-Suzuki-Tsuda	[Tsuda 2010]
$33, 33, 33, 321^{*1}$	Matrix type	[Kawakami 2010]
$42, 33, 33, 222$		[S 2012]
$51, 33, 222, 222$		[S 2012]
$51, 33, 33, 111111^{*1}$	Sasano	[Fuji et al. 2010]

*1 is extended to the $2n$-th order.

Some of Painlevé systems include rigid systems as particular solutions.

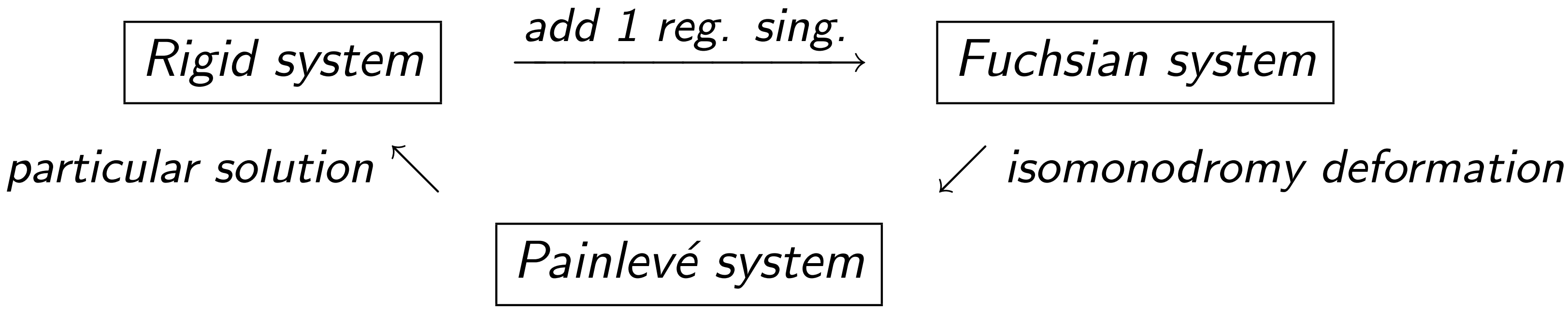
Painlevé system	Rigid system	HGF	Reference
$11, 11, 11, 11$	$11, 11, 11$	${}_2F_1$	[Fuchs 1907]
$11, 11, 11, 11, 11$	$21, 21, 21, 21$	P_3 (F_1)	[Fuchs 1907] [Garnier 1912]
$(21, 21, 21, 21, 21)$			
$21, 21, 111, 111$	$21, 111, 111$	${}_3F_2$	[S 2010] [Tsuda 2010]
$11, 11, 11, 11, 11, 11^{*2}$	$31, 31, 31, 31, 31$	P_4 (F_D)	[Garnier 1912]
$(31, 31, 31, 31, 31, 31)$			
$21, 21, 21, 21, 111$	$31, 31, 22, 211$	II_2^* (F_2)	[S 2012]
$(31, 31, 22, 22, 22)$			
$31, 31, 22, 22, 22$	$31, 22, 22, 22$	$P_{4,4}$ (F_4)	[S 2012]
$21, 111, 111, 111$	$211, 211, 211$	II_2	[S 2012]
$(31, 211, 211, 211)$			
$31, 22, 211, 1111$	$22, 211, 1111$	EO_4	[S 2012]
$31, 31, 1111, 1111^{*2}$	$31, 1111, 1111$	${}_4F_3$	[S 2010] [Tsuda 2010]

*2 is extended to the $2n$-th order.

We can find that any Fuchsian system which implies the Painlevé system is given by adding 1 regular singularity to the rigid system (the same as the HGDE!).

Let us recall the aim of this talk.

Problem 3.5. *For any rigid system, give a higher order Painlevé system which includes it as the following scheme:*



As is seen above, we have already solved for the cases of 2nd, 4th and 6th order. But we need a new construction method of canonical coordinates as the existing method seems to be not suitable for the case of higher order (future problem).

4 From Rigid to Painlevé

We try to formulate the 4th order Painlevé system of type $\{21, 21, 111, 111\}$ from the rigid system of type $\{21, 111, 111\}$ directly.
We consider a rigid system of type $\{21, 111, 111\}$

$$\frac{d}{dt}\mathbf{y} = A\,\mathbf{y} \quad (\mathbf{y} \in \mathbb{C}^3),$$

with

$$A = \frac{1}{1-t}\begin{bmatrix} \alpha_1 & \alpha_3 & \alpha_5 \\ \alpha_1 & \alpha_3 & \alpha_5 \\ \alpha_1 & \alpha_3 & \alpha_5 \end{bmatrix} + \frac{1}{t}\begin{bmatrix} -\alpha_2-\alpha_3-\alpha_4-\alpha_5 & \alpha_3 & \alpha_5 \\ 0 & -\alpha_4-\alpha_5 & \alpha_5 \\ 0 & 0 & 0 \end{bmatrix},$$

whose solution is described in terms of the generalized hypergeometric function ${}_3F_2$.

We give a transformation $A \to \widetilde{A}$ by putting

$$\alpha_1 \to \alpha_1 - q_1p_1 - q_2p_2 - \eta, \quad \alpha_3 \to \alpha_3 + q_1p_1, \quad \alpha_5 \to \alpha_5 + q_2p_2.$$

We also set

$$H = \begin{bmatrix} -q_1p_1 - q_2p_2 - \eta & p_1 & p_2 \end{bmatrix} \widetilde{A} \begin{bmatrix} 1 \\ q_1 \\ q_2 \end{bmatrix}.$$

Then H is equivalent to the Hamiltonian of the 4th order Painlevé system of type $\{21, 21, 111, 111\}$

$$\begin{aligned} H = {} & H_{\mathrm{VI}}^{-\alpha_1-\alpha_2-\alpha_3-\alpha_5,\alpha_2,\alpha_1+\alpha_5+\eta,\alpha_3\eta}(q_1,p_1) \\ & + H_{\mathrm{VI}}^{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5,\alpha_2+\alpha_4,\alpha_1+\alpha_3+\eta,\alpha_5\eta}(q_2,p_2) \\ & + \frac{(q_1-1)(q_2-t)\{(q_1p_1+\alpha_3)p_2 + p_1(q_2p_2+\alpha_5)\}}{t(t-1)}. \end{aligned}$$

Problem 4.1. *Is it possible to formulate similarly for the other rigid systems?*

5 Future Problem

We believe that the obtained result is a basis in order to construct a unified theory of higher order Painlevé systems and rigid systems.

We have just started to investigate the following problems.

① A new formulation method of Painlevé systems which include rigid systems

② Degenerations (Non-Fuchsian cases)

③ Discrete analogues or canonical quantizations

④ Application to the analysis both of Painlevé systems and rigid systems

Thank you.