Analytic Continuation of Solutions to Nonlinear Convolution Partial Differential Equations and its Application

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In my feeling, the structure of some convolution equations is very close to the structure of Maillet type theorem.

And so, the way of my research is;
(1) First, I consider a Maillet type theorem for a model equation.
(2) Then, I discuss a convolution equation corresponding to the model equation.
(3) After that, I apply the result for convolution equation to the problem of the summability.

And so, the plan of this talk is:

- **1. Maillet type theorem for a model equation**
- **2. Convolution PDEs correponding to the model equations**
- **3. Application to the summability of formal solutions**

1. Maillet type theorem for a model equation

1.1. A model equation

Let us us consider

$$P(t\partial_t, x)u = F\Big(t, x, \big\{(t\partial_t)^i \partial_x^\alpha u\big\}_{i+|\alpha|\le m}\Big). \tag{1.1}$$

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Set

$$Z = \{Z_{i,\alpha}\}_{i+|\alpha|\le m} \in \mathbb{C}^N,$$

and suppose:

1) l and m are integers with $0 \le l \le m$;
2) $P(\lambda, x) = \lambda^l + c_1(x)\lambda^{l-1} + \cdots + c_{l-1}(x)\lambda + c_l(x)$;
   and $c_i(x)$ are holomorphic near $x = 0$;
3) $F(t, x, Z)$ is a holomorphic function in a neighborhood
   of $(t, x, Z) = (0, 0, 0)$ satisfying

$$F(0, x, 0) \equiv 0 \quad \text{and} \quad \frac{\partial F}{\partial Z_{i,\alpha}}(0, x, 0) \equiv 0 \ \ (i + |\alpha| \le m)$$

1.2. Maillet type theorem

Theorem 1.1 (Maillet type theorem). If $\hat{u}(t,x) = \sum_{n\geq 1} u_n(x)t^n \in \mathcal{O}_R[[t]]$ is a formal solution of (1.1), there are $C > 0$ and $h > 0$ such that

$$|u_n(x)| \leq Ch^n n!^{s-1} \quad \text{on } D_R,\ n = 1, 2, \ldots.$$

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hold for some $s \geq 1$.

Moreover, s is calculated as follows.
By Taylor expansion in Z we have the expression

$$F(t,x,Z) = f(t,x) + \sum_{i+|\alpha|\leq m} a_{i,\alpha}(t,x)Z_{i,\alpha} + \sum_{|\nu|\geq 2} b_\nu(t,x)Z^\nu$$

where

$$f(t,x) = O(t^\mu) \quad \text{for some } \mu \geq 1,$$
$$a_{i,\alpha}(t,x) = O(t^{p_{i,\alpha}}) \quad \text{for some } p_{i,\alpha} \geq 1,$$
$$b_\nu(t,x) = O(t^{q_\nu}) \quad \text{for some } q_\nu \geq 0$$

1.3. Calculation of the index s

For $\nu = \{\nu_{i,\alpha}\}_{i+|\alpha|\leq m}$ we set

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We set

- $s_a = 1 + \max\Big[0, \max_{(i,\alpha)}\Big(\dfrac{i+|\alpha|-l}{p_{i,\alpha}}\Big)\Big];$
- $s_b = 1 + \max\Big[0, \max_{|\nu|\geq 2}\Big(\dfrac{m_\nu - l}{q_\nu + \mu(|\nu|-1)}\Big)\Big].$

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Then, s is given by

$$s = \max\{s_a, s_b\}.$$

2. Convolution PDEs correponding to the model equation

2.1. Some notations

Let $k > 0$, $I = (\theta_1, \theta_2)$ with $0 < |I| < 2\pi/k$, and $R > 0$.
We write $S_I = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\})\,;\, \theta_1 < \arg t < \theta_2\}$,

$$D_R = \{x \in \mathbb{C}^n\,;\, |x| < R\}.$$

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$D_R = \{x \in \mathbb{C}^n\,;\, |x| < R\}$.
For holomorphic functions $f(t,x)$ and $g(t,x)$ on $S_I \times D_R$, we define the k-convolution in t by

$$(f *_k g)(t,x) = \int_0^t f(\tau,x) g((t^k - \tau^k)^{1/k}, x) d\tau^k.$$

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For $(i, \alpha) \in \mathbb{N} \times \mathbb{N}^n$ we write

$$\mathscr{M}_{i,\alpha}[W] = \begin{cases} \dfrac{t^{k|\alpha|-k}}{\Gamma(|\alpha|)} *_k \left[(kt^k)^i W\right], & \text{if } |\alpha| > 0, \\ (kt^k)^i W, & \text{if } |\alpha| = 0. \end{cases}$$

2.2. Convolution PDEs

Let

- l and m are integers with $0 \le l \le m$;
- $P(\lambda, x) = \lambda^l + c_1(x)\lambda^{l-1} + \cdots + c_l(x)$;
- $c_i(x)$, $f(t, x)$, $a_{i,\alpha}(t, x)$ and $b_\nu(t, x)$ are all holomorphic functions on $S_I \times D_R$,

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- $c_i(x)$, $f(t,x)$, $a_{i,\alpha}(t,x)$ and $b_\nu(t,x)$ are all holomorphic functions on $S_I \times D_R$,

and let us consider

$$\text{(E)}\quad P(kt^k, x)u = f(t,x) + \sum_{i+|\alpha| \le m} a_{i,\alpha}(t,x) *_k \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right) + \sum_{|\nu| \ge 2} b_\nu(t,x) *_k \prod_{i+|\alpha| \le m}^{*_k} \left(\mathscr{M}_{i,\alpha}[\partial_x^\alpha u]\right)^{*_k \nu_{i,\alpha}}.$$

2.3. Note

We note that this equation (E) is nothing but the k-Borel transform of the equation

$$
\begin{aligned}
&P(t^{k+1}\partial_t, x)U \\
&= F(t,x) + \sum_{i+|\alpha|\le m} A_{i,\alpha}(t,x) \times t^{k|\alpha|}(t^{k+1}\partial_t)^i \partial_x^\alpha U \\
&\quad + \sum_{|\nu|\ge 2} B_\nu(t,x) \prod_{i+|\alpha|\le m} \left(t^{k|\alpha|}(t^{k+1}\partial_t)^i \partial_x^\alpha\right)^{\nu_{i,\alpha}}.
\end{aligned}
$$

2.4. Assumptions

1) $k > 0$ is an integer;
2) there are integers $\mu \geq 1$, $p_{i,\alpha} \geq 1$, $q_\nu \geq 1$ and positive constants $F > 0$, $A_{i,\alpha} > 0$, $B_\nu > 0$ such that

$$|f(t,x)| \leq \frac{F}{\Gamma(\mu/k)} |t|^{\mu-k} e^{c|t|^k} \quad \text{on } S_I \times D_R,$$

$$|a_{i,\alpha}(t,x)| \leq \frac{A_{i,\alpha}}{\Gamma(p_{i,\alpha}/k)} |t|^{p_{i,\alpha}-k} e^{c|t|^k} \quad \text{on } S_I \times D_R,$$

$$|b_\nu(t,x)| \leq \frac{B_\nu}{\Gamma(q_\nu/k)} |t|^{q_\nu-k} e^{c|t|^k} \quad \text{on } S_I \times D_R.$$

3) moreover, $\sum_{|\nu|\geq 2} B_\nu t^{q_\nu} X^{|\nu|}$ is convergent in a neighborhood of $(t, X) = (0, 0)$.

4) The roots $\lambda_1, \ldots, \lambda_l$ of $P(\lambda, 0) = 0$ satisfy

$$\lambda_i \in \mathbb{C} \setminus \overline{S_{kI}}, \quad i = 1, \ldots, l.$$

where $kI = (k\theta_1, k\theta_2)$.

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$$\boldsymbol{\lambda_i \in \mathbb{C} \setminus \overline{S_{kI}}, \quad i = 1, \ldots, l.}$$

where $kI = (k\theta_1, k\theta_2)$.

Set $S_I(\delta) = \{t \in S_I \, ; \, 0 < |t| < \delta\}$.

Then, as to the analytic continuation we have the following result.

2.6. Analytic continuation

Theorem 2.1 (Analytic continuation). If $u(t,x)$ is a local holomorphic solution of (E) on $S_I(\delta) \times D_R$ satisfying the estimate

$$|u(t,x)| \leq M_0 |t|^{\mu-k} \quad \text{on } S_I(\delta) \times D_R$$

for some $\delta > 0$ and $M_0 > 0$,

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for some $\delta > 0$ and $M_0 > 0$,
then it has an analytic continuation $u^*(t,x)$ on $S_I \times D_\rho$ which satisfies the estimate

$$|u^*(t,x)| \leq M |t|^{\mu-k} e^{b|t|^{\kappa}} \quad \text{on } S_I \times D_\rho$$

for some $M > 0$, $b > 0$, $\rho > 0$ and $\kappa > 0$ defined by

$$1/\kappa = 1/k - (s-1).$$

2.7. Calculation of the index s

For $x \in \mathbb{R}$ we write $[x]_+ = \max\{x, 0\}$.
For $\nu = \{\nu_{i,\alpha}\}_{i+|\alpha|\le m} \in \mathbb{N}^N$ we set

$$m_\nu = \max\{i + |\alpha| \,;\, \nu_{i,\alpha} > 0\},$$

$$\langle \nu \rangle_l = \sum_{i+|\alpha|\le m} [i + |\alpha| - l]_+ \nu_{i,\alpha}$$

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- $s_a = 1 + \max\left[0, \max_{(i,\alpha)} \left(\dfrac{[i + |\alpha| - l]_+}{p_{i,\alpha} + k[i + |\alpha| - l]_+} \right)\right],$
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Then, s is given by

$$s = \max\{s_a, s_b\}.$$

2.8. Some remarks

(1) In the theorem, we have supposed that k, μ, $p_{i,\alpha}$ and q_ν are integeres, but it is possible to show the same results in the case where k, μ, $p_{i,\alpha}$ and q_ν are rational numbers.

(2) The condition

$$\lambda_i \in \mathbb{C} \setminus \overline{S_{kI}}, \quad i = 1, \ldots, l$$

can be improved to

$$\lambda_i = 0 \quad \text{or} \quad \lambda_i \in \mathbb{C} \setminus \overline{S_{kI}}, \quad i = 1, \ldots, l.$$

3. Application to the summability of formal solutions (in progress)

3.1. Motivation

As is seen in the book of Gérard-Tahara, we know that the equation

$$(t\partial_t)^l u = H\Big(t, x, \{(t\partial_t)^i \partial_x^\alpha u\}_{i+|\alpha| \le m}\Big)$$

has a formal solution of the form

$$\hat{u}(t,x) = \sum_{mi+2m|j| \ge k} \varphi_{i,j,k}(x) t^{i+j_1\lambda_1(x)+\cdots+j_\mu\lambda_\mu(x)} (\log t)^k$$

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If $l < m$, this formal solution is not convergent in general.

Can we apply the summability methord to formal solutions of this type ? To answer this question is the main purpose of this research, though it is still in progress.

3.2. Equations

Let us consider the equation

$$F\Big(t, x, \{(t\partial_t)^i \partial_x^\alpha u\}_{i+|\alpha|\le m}\Big) = 0 \tag{3.1}$$

under the assumption:

(A_1) $F(t, x, Z)$ is a holomorphic function in a neighborhood of $(t, x, Z) = (0, 0, 0)$.

(A_2) $F(0, x, 0) \equiv 0$ in a neighborhood of $x = 0$.

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If we write

$$\Lambda = \{(i, \alpha) \in \mathbb{N} \times \mathbb{N}^n ; i + |\alpha| \le m\},$$
$$\Theta u = \{(t\partial_t)^i \partial_x^\alpha u\}_{(i,\alpha)\in\Lambda},$$

our equation is written as $F(t, x, \Theta u) = 0$.

3.3. Some notations

(1) For $\hat{u}(t,x) = \sum_{k\geq 1} u_k(t,x)$ and N we write

$$S_N(\hat{u}) = \sum_{1\leq k\leq N} u_k(t,x): \quad N\text{-partial sum.}$$

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(2) For a function $f(t,x) \in \mathcal{O}(S_I(r) \times D_R)$ we write $\|f(t)\|_R = O(|t|^A)$ if

$$\sup_{x\in D_R} |f(t,x)| = O(|t|^A) \ \text{(as } S_I \ni t \longrightarrow 0).$$

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$$\sup_{x\in D_R}|f(t,x)|=O(|t|^A) \ \ (\text{as } S_I\ni t\longrightarrow 0).$$

(3) For a bounded function $f(t,x)\in\mathcal{O}(S_I(r)\times D_R)$ we write

$$\gamma_t(f)=\sup\{k\geq 0\,;\,\|f(t)\|_R=O(|t|^k)\}.$$

This is an analogue of the order of zeros of $f(t,x)$ at $t=0$.

3.4. Meaning of a formal solution

Definition 3.1. A formal series

$$\hat{u}(t,x) = \sum_{n\geq 1} u_n(t,x) \in \sum_{n\geq 1} \mathcal{O}(S_I(r) \times D_R).$$

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is said to be a formal solution of (3.1), if the following two conditions are satisfied:

(1) there is a sequence $0 < q_1 < q_2 < \cdots$ such that $q_n \longrightarrow \infty$ (as $n \longrightarrow \infty$) and

$$\left\|(t\partial_t)^i \partial_x^\alpha u_n\right\|_R = O(|t|^{q_n}) \text{ for any } (i,\alpha) \in \Lambda.$$

(2) For any $A > 0$ there is an N_0 such that

$$\left\|F\big(t,x,\Theta S_N(\hat{u})\big)\right\|_R = O(|t|^A) \text{ for any } N \geq N_0.$$

3.5. Equivalence of two formal solutions

Definition 3.2. Let

$$\hat{u}(t,x) = \sum_{n \geq 1} u_n(t,x) \text{ and } \hat{w}(t,x) = \sum_{n \geq 1} w_n(t,x)$$

be two formal solutions of (3.1). We say that $\hat{u}(t,x)$ and $\hat{w}(t,x)$ are equivalent if they satisfy the following: for any $A > 0$ there is an N_0 such that

$$\left\| S_N(\hat{u}) - S_N(\hat{w}) \right\|_R = O(|t|^A) \text{ for any } N \geq N_0.$$

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We note that if $\hat{u}(t,x) = \sum_{n\geq 1} u_n(t,x)$ is a formal solution, then

$$\hat{w}(t,x) = \sum_{n\geq 1} \Big(u_n(t,x) + \text{flat function} \Big)$$

is also a formal solution which is equivalent to $\hat{u}(t,x)$.

3.6. Our setting

Let us return to our setting. We are considering

$$F\Big(t, x, \{(t\partial_t)^i \partial_x^\alpha u\}_{i+|\alpha|\le m}\Big) = 0 \tag{3.1}$$

under the conditions

(A_1) $F(t, x, Z)$ is a holomorphic function in a neighborhood of $(t, x, Z) = (0, 0, 0)$,

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Let I be an interval, $r > 0$ and $R > 0$. Suppose that a formal solution

$$\hat{u}(t, x) = \sum_{n\ge 1} u_n(t, x) \in \sum_{n\ge 1} \mathcal{O}(S_I(r) \times D_R).$$

of (2.1) is given, or it is already calculated.

3.7. Newton polygon

For the given formal solution $\hat{u}(t,x)$ we set

$$p_{i,\alpha} = \overline{\lim_{N\to\infty}}\, \gamma_t\Big(\frac{\partial F}{\partial Z_{i,\alpha}}(t,x,\Theta S_N(\hat{u}))\Big), \quad (i,\alpha) \in \Lambda :$$

then $p_{i,\alpha}$ is an analogue of the order of the zeros of $(\partial F/\partial Z_{i,\alpha})(t,x,\Theta\hat{u})$ at $t = 0$.

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then $p_{i,\alpha}$ is an analogue of the order of the zeros of $(\partial F/\partial Z_{i,\alpha})(t,x,\Theta\hat{u})$ at $t=0$.

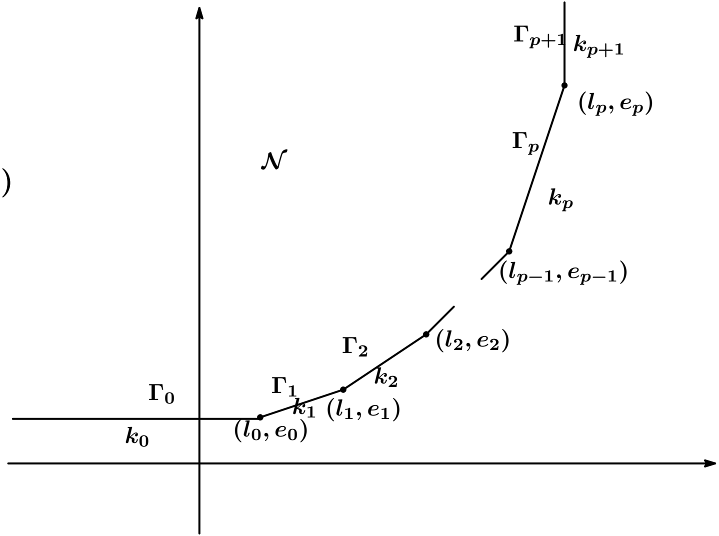
By using this, we define the Newton polygon $\mathcal{N}$ by

$$\mathcal{N} = \text{the convex hull of } \bigcup_{(i,\alpha)\in\Lambda} C(i+|\alpha|, p_{i,\alpha})$$

in $\mathbb{R}^2$, where $C(a,b) = \{(x,y) \in \mathbb{R}^2\,;\, x \le a, y \ge b\}$.

3.8. Picture of the Newton polygon

(3.2)



3.9. Important data in (3.2)

In the picture (3.2) we denoted:

$V = \{(l_0, e_0), \ \cdots, \ (l_p, e_p)\}$: the set of vertices of $\mathcal{N}$,

$\partial\mathcal{N} = \Gamma_0 \cup \cdots \cup \Gamma_{p+1}$: the boundary of $\mathcal{N}$,

k_i : the slope of Γ_i $(i = 0, 1, \ldots, p+1)$.

Then we have

$$k_0 = 0 < k_1 < k_2 < \cdots < k_p < k_{p+1} = \infty.$$

3.9. Important data in (3.2)

In the picture (3.2) we denoted:

$V = \{(l_0, e_0), \ \cdots, \ (l_p, e_p)\}$: the set of vertices of $\mathcal{N}$,
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Then we have

$$k_0 = 0 < k_1 < k_2 < \cdots < k_p < k_{p+1} = \infty.$$

We denote by $Int(\mathcal{N})$ the interior of $\mathcal{N}$, and we suppose:

(b_1) (l_p, e_p)(the last vertex) $= (m, p_{m,0})$

(b_2) all vertices are rational points.

(b_3) $(i, \alpha) \in \Lambda$ and $|\alpha| > 0$

$$\Longrightarrow (i + |\alpha|, p_{i,\alpha}) \in Int(\mathcal{N}).$$

3.10. Further assumptions

In addition to (b_1)-(b_3) we suppose:

(b_4) If $(i, p_{i,0}) \in \partial\mathcal{N}$, there are N_0, $\varphi_{i,0}(x) \in \mathcal{O}_R$ and $\epsilon > 0$ such that for any $N \geq N_0$ we have the expression:

$$\frac{\partial F}{\partial Z_{i,0}}(t, x, \Theta S_N(\hat{u})) = \varphi_{i,0}(x) t^{p_{i,0}} + O(|t|^{p_{i,0}+\epsilon})$$

(as $S_I \ni t \longrightarrow 0$).

(b_5) If $(i, p_{i,0}) \in V$, then $\varphi_{i,0}(0) \neq 0$.

(b_6) The interval $I = (\theta_1, \theta_2)$ satisfies

$$|I| = \theta_2 - \theta_1 > \frac{\pi}{k_1}.$$

3.11. Multisummability

Theorem 3.3. The equation (3.1) has a new formal solution

$$\hat{w}(t,x) = \sum_{n \geq 1} w_n(t,x) \in \sum_{n \geq 1} \mathcal{O}(S_I(\delta) \times D_R)$$

which is equivalent to the given formal solution $\hat{u}(t,x)$ and is multisummable in a suitable direction,

Theorem 3.3. The equation (3.1) has a new formal solution

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which is equivalent to the given formal solution $\hat{u}(t,x)$ and is multisummable in a suitable direction, that is, there are

- interval $K \subset I$ with $|K| > \pi/k_p$, $\delta_1 > 0$, $R_1 > 0$,
- holomorphic solution $w(t,x)$ of (3.1) on $S_K(\delta_1) \times D_{R_1}$
- an integer $L > 0$ and constants $C > 0$ and $h > 0$

such that

$$\Bigl| w(t,x) - \sum_{n=1}^{N-1} w_n(t,x) \Bigr| \leq C h^N \Gamma\Bigl(\frac{N}{L k_1}\Bigr) |t|^{N/L}$$

on $S_K(\delta_1) \times D_{R_1}$ for any $N \geq 1$.

3.12. Sketch of Proof

Let

$$\hat{u}(t,x) = \sum_{n \geq 1} u_n(t,x)$$

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Step 1. We take sufficiently large N and q, and set

$$u(t,x) = \sum_{n=1}^{N} u_n(t,x) + t^q w(t,x).$$

Then, our equation is written as

$$F\big(t, x, \Theta S_N(\hat{u}) + \Theta t^q w\big) = 0. \tag{3.3}$$

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Then, our equation is written as

$$F\big(t, x, \Theta S_N(\hat{u}) + \Theta t^q w\big) = 0. \tag{3.3}$$

By using the equation (3.3) we find a new formal solution

$$\hat{w}(t,x) = \sum_{n\geq 1} w_n(t,x).$$

3.13. Sketch of Proof (continued)

Step 2. **We show Gevrey type estimates of this formal soluion $\hat{w}(t, x)$.**

3.13. Sketch of Proof (continued)

Step 2. We show Gevrey type estimates of this formal soluion $\hat{w}(t,x)$.

Step 3. We define the formal k_1-Borel transform of $\hat{w}(t,x)$ by

$$\hat{\mathcal{B}}_{k_1}[\hat{w}] = \sum_{n\geq 1} \mathcal{B}_{k_1}[w_n].$$

Then, we can reduce the problem to the problem of analytic continuation of solutions of convolution PDEs.

Thank you for your attention.