Asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation

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Today is the birthday of B. Riemann and the speaker.

1. Riemann-Hilbert problem (RHP)

BOUNDARY VALUE PROBLEM IN THE COMPLEX PLANE

Γ: oriented contour (the left-hand is the $+$ side).

$m(z)$: unknown matrix, holomorphic in $\mathbb{C} \setminus \Gamma$

Examples: 1. $\Gamma = \mathbb{R}$, $m(z)$ holo. in $\pm \mathrm{Im}\, z > 0$.

2. $\Gamma = \{|z| = 1\}$, $m(z)$: holo. in $|z| \neq 1$.

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m_+, m_-: boundary values on Γ from the $\pm$ sides

RHP: $m_+ = m_- v$ on Γ (v : **the jump matrix**)

We often neglect to mention the normalization condition $m(z) \to I$ as $z \to \infty$.

2. RHPs behave like integrals

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contour deformation

New contour, unknown, jump matrix.
The original RHP $\Leftrightarrow$ new RHP.

continuity

The mapping $v \mapsto m$ is continuous.

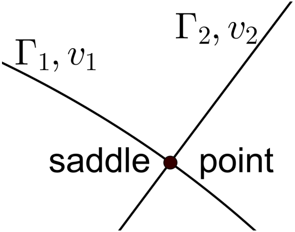
deletion of a part of the contour

1. If $v = I$ on $\hat{\Gamma} \subset \Gamma$ (no jump there),
$$m[\text{original}] = m[\text{with } \hat{\Gamma} \text{ deleted}]$$
2. If $v \approx I$ on $\hat{\Gamma}$, $m[\text{original}] \approx m[\text{with } \hat{\Gamma} \text{ deleted}]$

3. Nonlinear steepest descent (Deift-Zhou '93)

integrals	RHP
steepest descent	nonlinear steepest descent

Study $m_+ = m_- v$ on Γ when v involves $\exp[\pm it\psi(z)]$.



Deform Γ into $\Gamma_1 \cup \Gamma_2$. jump matrix v_j on Γ_j.

Assume: as $t \to \infty$, $v_j \to I$ on $\Gamma_j \setminus \{\text{saddle point}\}$.

Then $m(z)$ is almost determined by the value of v at the saddle point.

4. Inverse scattering for NLS and RHP

$$\boxed{iu_t + u_{xx} - 2|u|^2 u = 0 \cdots (\mathrm{NLS})}$$

$r(z,t)$: reflection coefficient

$$v_1(z) := \begin{bmatrix} 1 - |r(z,0)|^2 & -e^{-2it\psi_1}\overline{r(z,0)} \\ e^{2it\psi_1} r(z,0) & 1 \end{bmatrix}, \quad \psi_1 := 2z^2 + \frac{xz}{t}$$

$$m_+(z) = m_-(z) v_1(z) \quad \text{on } \mathbb{R},$$
$$m(z) \to I \; (z \to \infty)$$

$$u(x,t) = 2i \lim_{z\to\infty} z\, m(z;x,t)_{12} \text{ (reconstruction formula)}$$

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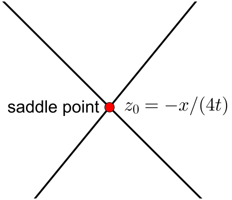
$$\boxed{u(x,0) \underset{x}{\mapsto} r(z) = r(z,0) \mapsto \mathrm{RHP} \mapsto m \mapsto u(x,t)}$$

5. Asymptotics of NLS

1. Zakhalov-Manakov '76: formal calculation
2. Deift-Its-Zhou '93: proof by *nonlinear steepest descent*

RHP involving $\exp(it\psi_1)$.
$\psi_1 = 2z^2 + xz/t$.



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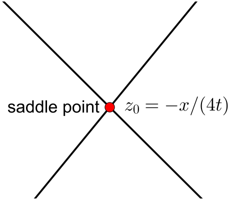

$z_0 = -x/(4t)$ is the only saddle point of ψ_1.

$$u(x,t) \sim \alpha(z_0) t^{-1/2} \exp\left(4itz_0^2 - i\nu(z_0)\log 8t\right)$$

6. Integrable Discrete NLS (IDNLS)

Ablowitz-Ladik ('75) introduced

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0 \quad \text{(IDNLS)}$$

R_n is asymptotically (Y. 2014, 2015)

1. $|n|/t < 2$ ('timelike' region)
   Sum of two terms, each being $t^{-1/2}\times$(oscillatory factor)
2. $|n|/t \approx 2$ ('light cone')
   $t^{-1/3}\times$(oscillatory factor)
   coefficient written in terms of a sol. of the Painlevé II eq.
   $$u'' - su(s) - 2u^3(s) = 0$$
3. $|n|/t > 2$ ('spacelike' region)
   $O(n^{-j})$ as $|n| \to \infty$

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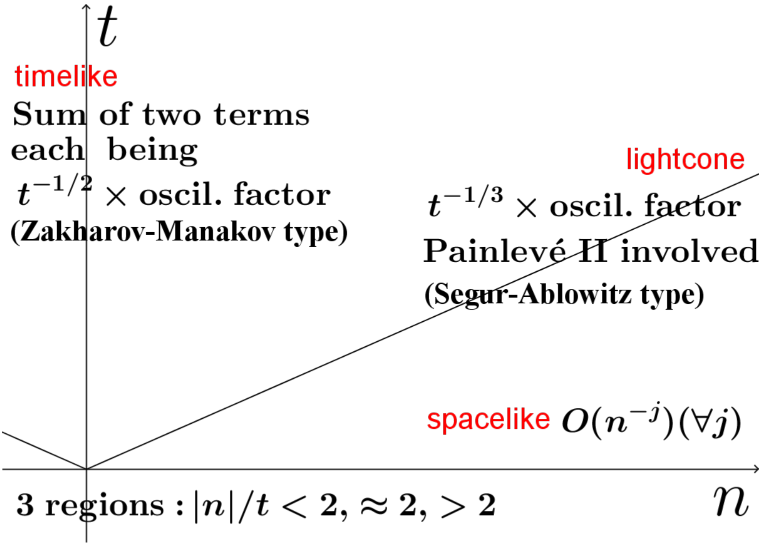
cf. formal calculation by Novokshënov about the focusing, solitonless case

7. Asymptotics: three regions



8. IDNLS and its Lax pair

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0 \quad \text{(IDNLS)}$$

....

n- and t-parts

$$X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$$

$$\frac{d}{dt}X_n = \Big[\text{a complicated matrix}\Big] X_n$$

(IDNLS) is the compatibility condition.

9. Reflection coefficient

$$X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$$

Ψ_n: holo. sol. in $|z| > 1$, continuous in $|z| \geq 1$,

Ψ_n^*: holo. sol. in $|z| < 1$, continuous in $|z| \leq 1$,

$$\Psi_n \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Psi_n^* \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{as } n \to \infty.$$

The reflection coefficient r is defined by :

$$\underbrace{r\Psi_n}_{\text{reflection}} + \underbrace{\Psi_n^*}_{\text{incidence}} \sim \text{const.} z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (n \to -\infty).$$

$r(z,t) = r(z)\exp\left(it(z - z^{-1})^2\right)$, where $r(z) = r(z,0)$.

10. RHP

$$m_+(z) = m_-(z)v_2(z) \text{ on } \underline{|z| = 1},$$

$$m(z) \to I \text{ as } z \to \infty,$$

$$v_2(z) = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2it\psi_2}\bar{r}(z) \\ e^{2it\psi_2}r(z) & 1 \end{bmatrix} \text{ jump matrix}$$

$$\psi_2 = \frac{1}{2}(z - z^{-1})^2 + \frac{in}{t}\log z$$

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Reconstruction formula $R_n(t) = -\left. \frac{d}{dz} m(z)_{21} \right|_{z=0}$

RHP gives $\{R_n(t)\}$. IVP solved!

11. Saddle points and asymptotics

$$\psi_2(z) = \psi_2(z, n, t) = \frac{1}{2}(z - z^{-1})^2 + \frac{in}{t} \log z.$$

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The geometry of its saddle points determine the asymptotic behavior of $R_n(t)$.

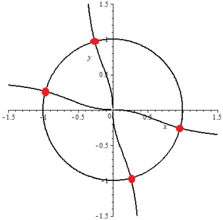
- $|n|/t < 2$: four saddle points on $|z|=1$ (simple zeros of ψ_2')
- $|n|/t = 2$: saddle points coalesce (two double zeros of ψ_2')
- $|n|/t > 2$: four saddle points *off* $|z| = 1$

Different behaviors in different regions.

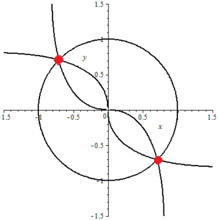
12. Geometry

The curve $\boxed{\operatorname{Im}\psi_2(z) = 0}$ and the saddle points or the stationary points of higher order.

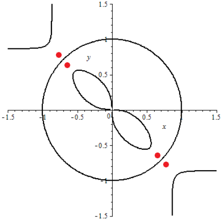
$|n|/t < 2$(saddle)



$|n|/t = 2$



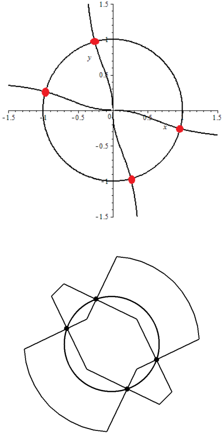
$|n|/t > 2$(saddle)



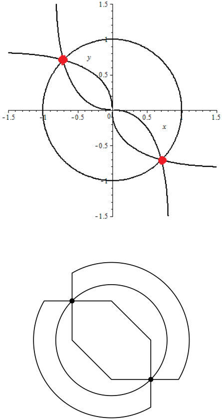
13. contour deformation

The curve $\mathrm{Im}\psi_2(z) = 0$ and the new (steepest descent) contour

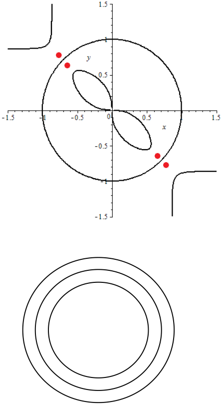
$|n|/t < 2$(saddle)



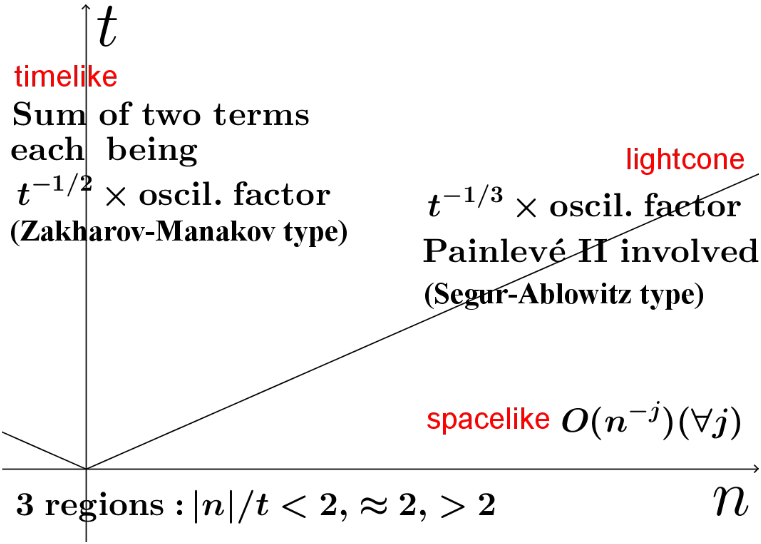
$|n|/t = 2$



$|n|/t > 2$(saddle)



14. Different behaviors in different regions



Thank you very much!

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Research of the focusing case is now in progess.