LINEAR MULTI-VARIABLE POLYLOGARITHMS

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Plan

Classical polylogarithm

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$$\mathrm{Li}_2(t) := \sum_{n>0} \frac{t^n}{n^2}, \quad |t| < 1, \tag{1}$$

has been defined and extensively studied by Euler (mainly, but not only, in the article cited below).

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As $\mathrm{re}\, s > 1$, function Li_s is well defined on a closure $\overline{D}$ of the unit disc D and we have $\mathrm{Li}_s(1) = \zeta(s)$, where ζ is the famous Riemann zeta function.

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It follows, that the polylogarithmic function satisfies differential equation

$$\theta^s \mathrm{Li}_s(t) = \frac{t}{1-t}, \quad |t| < 1 \tag{3}$$

or, equivalently,

$$(1-t)\partial_t \theta_t^{s-1} \mathrm{Li}_s(t) = 1, \quad |t| < 1. \tag{4}$$

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This differential equation, (3) or (4), can be engaged to analiticaly continue Li_s on $\mathbb{C}\backslash\{0,1\}$, i.e. $\mathbb{C}P^1\backslash\{0,1,\infty\}$.

Classical polylogarithm. Integral

Define the operator T as follows. Let $Tf(t) := \int_0^t f(\tau)/\tau d\tau$, on $t\mathbb{C}[t]$. In more general context, we understand T as a linear extension of the above oerator.

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Classical polylogarithm - modern point of view

Equation (3) or (4) can be used to define *Fuchsian connection* ∇_E, acting on the sections of a vector bundle $E \to \mathbb{C}P^1 \backslash \{0, 1, \infty\}$ over a 3-*pointed algebraic line*.

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Euler-Gauss hypergeometric function

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\begin{aligned}
{}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t\right) &= \sum_{n \geqslant 0} \frac{(u)_n (v)_n}{(w)_n} \frac{t^n}{n!} \qquad (6) \\
&= 1 + \frac{u \cdot v}{w} t + \frac{u(u+1) \cdot v(v+1)}{w(w+1)} \frac{t^2}{2!} + O(t^3),
\end{aligned}
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where $|t| < 1$ and $(x)_n := x(x+1)...(x+n-1)$ is the Pochhammer function.

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where $|t| < 1$ and $(x)_n := x(x+1)...(x+n-1)$ is the Pochhammer function.

It has been introduced by Euler and studied by the leading matematicians of the XIX and the beginning of XX century, including Gauss, Riemann (monodromy, P-function, Riemann surfaces), Kummer (bases of solutions, special values), Shwarz (Shwarz list) and others.

General classical hypergeometric function

One can easily generalize the classical Euler-Gauss hypergeometric function, by the series ($0 < p, q \in \mathbb{Z}$ are parameters, such that $p \leqslant q + 1$)

$$ {}_pF_q\left(\begin{matrix} u_1, u_2, \ldots, u_p \\ w_1, w_2, \ldots, w_q \end{matrix} \;\middle|\; t \right) = \sum_{n \geqslant 0} \frac{(u_1)_n (u_2)_n \ldots (u_p)_n}{(w_1)_n (w_2)_n \ldots (w_q)_n} \frac{t^n}{n!}, \tag{7} $$

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If $p < q + 1$, then function (7) is called *confluent* and if $p = q + 1$, then it is called *balanced*.

General classical hypergeometric function and Li_s

For integer $s > 0$, we have

$$t\,{}_{s+1}F_s\left(\begin{matrix}1, 1, \ldots, 1\\ 2, 2, \ldots, 2\end{matrix}\;\middle|\; t\right) = \mathrm{Li}_s(t) \tag{8}$$

and

$$t\,{}_{s}F_{s-1}\left(\begin{matrix}2, 2, \ldots, 2\\ 1, 1, \ldots, 1\end{matrix}\;\middle|\; t\right) = \mathrm{Li}_{-s}(t). \tag{9}$$

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Hypergeometric differential equation

We introduce the following operators: the multiplication operator $f(t) \mapsto tf(t)$, wich we simply denote by t, differential operator $\partial_t := d/dt$ and - allready previously mentioned - the Euler operator $\theta_t = t\partial t$.

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$$(\theta_t + u)(\theta_t + v)\,{}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t\right) = (\theta_t + w)\partial_t\,{}_2F_1\left(\begin{matrix} u+1, v \\ w \end{matrix} \,\middle|\, t\right). \quad (10)$$

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Or in equivalent form:

$$\left\{ t(t-1)\partial_t^2 + ((u+v+1)t - w)\partial_t + uv \right\} {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \;\middle|\; t\right) = 0. \quad (11)$$

General classical hypergeometric equation

The analog of the Euler-Gauss hypergeometric equation for general classical hypergeometric function can be written as

$$t\,P(\theta_t)\,{}_pF_q \;=\; Q(\theta_t)\,{}_pF_q, \tag{12}$$

where

$$\begin{aligned} P(x) &= (x+u_1)(x+u_2)...(x+u_p) \\ Q(x) &= (x+w_1-1)(x+w_2-1)...(x+w_q-1). \end{aligned}$$

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From the above differential equation, one can restore the classical hypergeometric series, as a particular solution. Note that for particular set of parameters, $a = p-1$, $u_i = 1$ and $w_j = 2$, we get the operator associated to polylogarithm.

Integral representations

Hypergeometric functions can be represented by several types of integrals.

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Hypergeometric functions can be represented by several types of integrals. One of them is the Euler representation:

$$\frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}\,{}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix}\;\middle|\; t\right) = \int_0^1 x^{v-1}(1-x)^{w-v-1}(1-tx)^{-u}\,dx.$$

Integral representations

The other useful formula is the Mellin-Barnes integral:

$$\begin{aligned} &\frac{\Gamma(u)\Gamma(v)}{\Gamma(w)}\, {}_2F_1\left(\left.\begin{matrix} u, v \\ w \end{matrix}\,\right|\, t\right) \\ = \quad &\frac{1}{2\pi i}\int_C \frac{\Gamma(u+s)\Gamma(v+s)}{\Gamma(w+s)}\,\Gamma(-s)(-t)^s\, ds, \end{aligned} \tag{13}$$

where the contour C is a line from $-i\infty + s_0$ to $-i\infty + s_0$, for some $s_0 \in \mathbb{R}$, separating poles of $\Gamma(-s)$ from the poles of the other Γ-factors.

Integral representations: particular example

The Euler integral formula can be used, by putting $-u =: \lambda := v$ and $w = 1 = t$, to deliver the following representation:

$$\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(1)} \, {}_2F_1\left(\begin{matrix} -\lambda, \lambda \\ 1 \end{matrix} \;\middle|\; 1\right) = \int_0^1 x^{\lambda-1} dx,$$

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equivalent to

$${}_2F_1\left(\begin{matrix}-\lambda, \lambda \\ 1\end{matrix}\;\middle|\; 1\right) = \frac{1}{\Gamma(1+\lambda)\Gamma(1-\lambda)} = \frac{\sin \pi\lambda}{\pi\lambda}. \tag{14}$$

Motivation

In what follows, we will define certain multi-variable generalization of the polylogarithm and estabilish its relation to multiple-zeta values, by analogy of relation of classical polylogarithm to the Riemann-zeta function.

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In what follows, we will define certain multi-variable generalization of the polylogarithm and estabilish its relation to multiple-zeta values, by analogy of relation of classical polylogarithm to the Riemann-zeta function. Later on we will relate theese functions to general multi-variable hypergeometric functions.

Shintani zeta-function

Shintani zeta function (or Shintani L-function) is a generalization of the Riemann zeta function. They were first studied by Takuro Shintani (1976). They include Hurwitz zeta functions, Barnes zeta functions, and Witten zeta functions as special cases.

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The Shintani zeta function of $s = (s_1, ..., s_r) \in \Omega \subset \mathbb{C}^r$ is given by

$$\zeta_S(s) \ := \sum_{n_1,\ldots,n_m \geqslant 0} \frac{1}{l_1^{s_1} \cdots l_r^{s_r}}, \tag{15}$$

where each l_j is an affine function of $n = (n_1, ..., n_m) \in \mathbb{N}^m$. The special case when $r = 1$ is the Barnes zeta function.

Multiple ζ function

Definition

The multiple zeta function is defined by the series

$$\sum_{n_p > \ldots > n_2 > n_1 > 0} n_1^{-s_1} n_2^{-s_2} \ldots n_p^{-s_p} := \zeta(s_1, s_2, \ldots, s_p), \tag{16}$$

whenever (16) converges. Number p is called depth, and $|s| := s_1 + s_2 + \ldots + s_p$ - weight of $\zeta(s_1, s_2, \ldots, s_p)$. Multiple zeta values (in short MZV), are values of multiple zeta function at integral points.

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To simplify nontation, one writes $(\{s_1, \ldots, s_q\}^n)$, meaning $(s_1, \ldots, s_q, s_1, \ldots, s_q, \ldots, s_1, \ldots, s_q)$, where $(s_1, \ldots, s_q)$ is repeated n times.

Multiple ζ function and their generalisations

If $p = 1$, then multiple zeta function is simply the Riemann zeta function

$$\sum_{n>0} n^{-s} = \zeta(s). \tag{17}$$

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Multiple zeta values apeared for the first time in Euler's *Meditationes circa singulare serierum genus* (1775), where he found the following formula relating Multiple Zeta Values to 'single' ones:

$$\sum_{n>0} \frac{H_n}{(n+1)^2} = \zeta(2,1) = \zeta(3) = \sum_{n>0} \frac{1}{n^3}, \tag{18}$$

where H_m is the m-th harmonic number.

Relations between multiple zeta values

MZV satisfy a lot of relations. For example

$$
\begin{aligned}
\zeta(r)\zeta(s) &= \sum_{m,n>0} m^{-r} n^{-s} \\
&= \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0} \right) m^{-r} n^{-s} \\
&= \zeta(r,s) + \zeta(s,r) + \zeta(r+s). \qquad (19)
\end{aligned}
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$$\begin{aligned}\zeta(r)\zeta(s) &= \sum_{m,n>0} m^{-r} n^{-s} \\ &= \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0}\right) m^{-r} n^{-s} \\ &= \zeta(r,s) + \zeta(s,r) + \zeta(r+s). \qquad (19)\end{aligned}$$

Other nontrivial relations can be obtained from the Drinfeld-Kontsevich integral.

Drinfeld-Kontsevich integral

The Drinfeld-Kontsevich integral for (multiple) polylogarithm function is obtained by applicaion of iterated formal inverse of differential operator $(1-t)/t\,\theta^s$, (multiplicatively) anihilating Li_s.

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$$\mathrm{Li}_{s_1,\ldots,s_r}(t) := \int_\gamma \omega, \qquad \text{where } \gamma = \{0 < t_{s_1+\ldots s_r} < \ldots < t_1 < t < 1\} \tag{20}$$

is the standard simplex in $\mathbb{R}^{s_1+s_2+\ldots s_r}$ and where ω is the differential form obtained by iteration of dt/t and $dt/(1-t)$:

$$\omega := \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_{s_1-1}}{t_{s_1-1}} \wedge \frac{dt_{s_1}}{1-t_{s_1}} \wedge \ldots \wedge \frac{dt_{s_1+\ldots+s_r-1}}{t_{s_1\ldots+s_r-1}} \wedge \frac{dt_{s_1+\ldots+s_r}}{1-t_{s_1+\ldots+s_r}}. \tag{21}$$

Drinfeld-Kontsevich integral

To shorten the notation, we write

$$\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_{s_i}}{t_{s_i}} := \left(\frac{dt}{t}\right)^{\circ i}.$$

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$$\omega = \left(\frac{dt}{t}\right)^{\circ s_1} \wedge \frac{dt_{s_1-1}}{1-t_{s_1}} \dots \left(\frac{dt}{t}\right)^{\circ s_r - 1} \wedge \frac{dt_{s_1+\dots+s_r}}{1-t_{s_1+\dots+s_r}}.$$

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Since $\int_\gamma \omega$ is obtained by application of the iterated inverse of differential operators associated to polylogarithm function, it follows immidiatelly that

$$\frac{1-t}{t}\theta^{s_r}\frac{1-t}{t}\theta^{s_{r-1}}\ldots\frac{1-t}{t}\theta^{s_1}\mathrm{Li}_{s_1,\ldots,s_r}(t) = 1. \tag{22}$$

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Denote the above by $T_{s_r}\ldots T_{s_1}$. Now, meaning that $T_{s_r}\ldots T_{s_1}\int_\gamma \omega = 1$, we write $\int_\gamma \omega = (T_{s_r}\ldots T_{s_1})^{-1}$ and that the operator is applied to the constant function, equal everywhere to 1.

Linear multi-variable polylogarithm

Let A denote matrix, corresponding to an affine map $A : \mathbb{R}^k \to \mathbb{R}^n$, of maximal rank. We denote the image $A(\mathbb{Z}^k)$ by L and $A(\mathbb{N}^k)$ by L_+.

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$$\mathrm{Gon}(s, t) := \sum_{0 < n_1 < \ldots < n_r} \frac{t_1^{n_1} \ldots t_1^{n_r}}{n_1^{s_1} \ldots n_r^{s_r}}, \qquad |t_i| \leqslant 1,\ s \in \mathbb{N}^r > 1,\ s_1 > 1. \tag{26}$$

Linear multi-variable polylogarithm of an integral latice

In all arithmetic aplications, matrix A is defined over $\mathbb{Z}$.

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Example: Mordell-Tornheim polylogarithm

Consider the following special case of Shintani zeta-function, the, so calld *Mordell-Tornheim polylogarithmic series*:

$$\mathrm{Li}_{\mathrm{MT},s_1,s_2,s_3}(t_1,t_2,t_3) := \sum_{m,n>0} \frac{t_1^m t_2^n t_3^{m+n}}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{27}$$

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In particular, puting $x = t_1 t_2$ and $y = t_2 t_3$, we get

$$\mathrm{Li}_{\mathrm{MT},1,1,1}(t_1,t_2,t_3) = \sum_{m,n>0} \frac{x^m y^n}{mn(m+n)}. \tag{29}$$

and $\zeta_{\mathrm{MT}}(1,1,1) = 2\zeta(3)$.

Prologue. The simplest two-variable hyp. series

If we multiply two hypergeometric series, ${}_2F_1(u, v, w \,|x)$ and ${}_2F_1(u', v', w' \,|y)$, then we end up with the following series

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Because of obvious reasons, we call the above series *reducible*. Identities between binomial coefficients (and Pochhammer functions) led matematicians to the *irreducible* hypergeometric series. Before we will give several examples, note the analogy with the following *reducible* case of Shintani zeta-function:

$$ \zeta(2)^2 = \sum_{m,n>0} m^{-2} m^{-2} = \zeta(4) + \sum_{m,n>0} m^{-2}(m+n)^{-2}. \qquad (30) $$

Appell functions

In 1880 P. Appell defined the following list of hypergeometric functions of two variables:

$$F_1\left(\begin{matrix} u, v_1, v_2 \\ w \end{matrix} \,\middle|\, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u)_{m+n}(v_1)_m(v_1)_n}{(w)_{m+n}} \frac{x^m y^n}{m!n!}, \tag{31}$$

$$F_2\left(\begin{matrix} u, v_1, v_2 \\ w_1, w_2 \end{matrix} \,\middle|\, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u)_{m+n}(v_1)_m(v_1)_n}{(w_1)_m(w_2)_n} \frac{x^m y^n}{m!n!}, \tag{32}$$

$$F_3\left(\begin{matrix} u_1, u_2, v_1, v_2 \\ w \end{matrix} \,\middle|\, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u_1)_m(u_2)_n(v_1)_m(v_1)_n}{(w)_{m+n}} \frac{x^m y^n}{m!n!}, \tag{33}$$

$$F_4\left(\begin{matrix} u, v \\ w_1, w_2 \end{matrix} \,\middle|\, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u)_{m+n}(v)_{m+n}}{(w_1)_m(w_2)_n} \frac{x^m y^n}{m!n!}. \tag{34}$$

Appell and Horn functions

Series defining functions F_1, F_2, F_3, F_4 converge in open subsets of $\mathbb{C}^2$.

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Series defining functions F_1, F_2, F_3, F_4 converge in open subsets of $\mathbb{C}^2$. In addition to the list of four Appell functions, there are 10 other balanced hypergeometric series and futher 20 confluent series, that have been enumerated by Horn (1931) and corrected by Borngässer (1933), in his dissertation, written in Darmstadt.

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Mordell-Tornheim hypergeometric series

Recall, that $t^{-1}\mathrm{Li}_2(t)$ is a hypergeometric function.

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Recall, that $t^{-1}\mathrm{Li}_2(t)$ is a hypergeometric function. We define the third-order two-variable generalization of Appell series:

$$F\left(\begin{matrix}\alpha, \beta_1, \beta_1', \beta_2, \beta_2' \\ \eta, \eta', \gamma\end{matrix}\,\middle|\, x, y\right) := \sum_{m,n\geqslant 0} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_1')_n(\beta_2)_m(\beta_2')_n}{(\eta)_m(\eta')_n(\gamma)_{m+n}\, m!\, n!}\, x^m y^n. \tag{35}$$

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If we put $(\alpha,\beta_1,\beta_1',\beta_2,\beta_2',\eta,\eta',\gamma) = (2,1,1,1,1,2,2,3)$, then we obtain

$$xy\, F\left(\begin{matrix}2,1,1,1,1\\ 2,2,3\end{matrix}\,\middle|\, x,y\right) = 2\sum_{m,n>0}^{\infty} \frac{x^m y^n}{mn(m+n)}\,. \tag{36}$$

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$$F\left(\begin{matrix}2,1,1,1,1\\2,2,3;\end{matrix}\,\middle|\,x,y\right) = 2\sum_{m,n>0}^{\infty}\frac{1}{mn(m+n)} = 2\zeta_{\mathrm{MT}}(1,1,1). \qquad (38)$$

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Thus, we found a hypergeometric function, thats specialization gives $\mathrm{Li}_{\mathrm{MT},1,1,1}$, in analogy with $t\,{}_3F_2(1,1,1,2,2\,|\,t) = \mathrm{Li}_2(t)$ and ${}_3F_2(1,1,1,2,2\,|\,1) = \zeta(2)$.

Differential equations and integral representations

All Appell and Horn (as well as Luricella) functions satisfy meromorphic differential equations. In addition they can be representad by integrals.

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All Appell and Horn (as well as Luricella) functions satisfy meromorphic differential equations. In addition they can be representad by integrals. For example, the Euler-type integral for F_2 is

$$\begin{aligned}
& \frac{\Gamma(v_1)\Gamma(v_2)\Gamma(w_1 - v_1)\Gamma(w_2 - v_2)}{\Gamma(w_1)\Gamma(w_2)} F_2 \left(\begin{matrix} u, v_1, v_2 \\ w_1, w_2 \end{matrix} \middle| \, x, y \right) \\
= \quad & \int_0^1 \int_0^1 t_1^{v_1 - 1} t_2^{v_2 - 1} (1 - t_1)^{w_1 - v_1 - 1} (1 - t_2)^{w_2 - v_2 - 1} \\
\times \quad & (1 - t_1 x - t_2 y)^{-u} \, dt_1 \, dt_2 .
\end{aligned}$$

If we write

$$F\left(\begin{matrix}\alpha,\beta_1,\beta_1',\beta_2,\beta_2'\\ \eta,\eta',\gamma\end{matrix}\,\middle|\, x,y\right) = \sum_{m,n\geqslant 0}^{\infty} c_{m,n}x^m y^n, \tag{39}$$

then the shift of the coefficients $c_{m+1,n}$ and $c_{m,n+1}$ leads to

$$c_{m+1,n} = \frac{(\alpha+m+n)(\beta_1+m)(\beta_2+m)}{(\eta+m)(1+m)(\gamma+m+n)}c_{m,n}, \tag{40}$$

and

$$c_{m,n+1} = \frac{(\alpha+m+n)(\beta_1'+n)(\beta_2'+n)}{(\eta'+n)(1+n)(\gamma+m+n)}c_{m,n}. \tag{41}$$

MT hpergeometric series and its Differential system

Let $\theta_x := x\partial/\partial x$ and $\theta_y := y\partial/\partial y$. From the above relations, replacing multiplication by m and n by Euler operators θ_x and θ_y, respectively, we get the following system of differential equations:

$$\begin{aligned}
&[x(\theta_x+\theta_y+\alpha)(\theta_x+\beta_1)(\theta_x+\beta_2)- \\
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In particular, for specialized parameters, we obtain the system associated with $\mathrm{Li}_{\mathrm{MT},1,1,1}(t)$.

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Let $\theta_x := x\partial/\partial x$ and $\theta_y := y\partial/\partial y$. From the above relations, replacing multiplication by m and n by Euler operators θ_x and θ_y, respectively, we get the following system of differential equations:

$$[x(\theta_x+\theta_y+\alpha)(\theta_x+\beta_1)(\theta_x+\beta_2)- \theta_x(\theta_x+\eta)(\theta_x+\theta_y+\gamma-1)].F = 0 \tag{42}$$

$$[y(\theta_x+\theta_y+\alpha)(\theta_y+\beta'_1)(\theta_y+\beta'_2)- \theta_y(\theta_y+\eta')(\theta_x+\theta_y+\gamma-1)].F = 0. \tag{43}$$

In particular, for specialized parameters, we obtain the system associated with $\mathrm{Li}_{\mathrm{MT},1,1,1}(t)$. We denote operator associated to this system by $T_{\mathrm{MT},1,1,1}$.

GKZ hypergeometric system

Let A denote $d \times n$ matrix of rank d with coefficients in $\mathbb{Z}$.

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Definition

Let $u \in \mathbb{C}^d$. Define

$$I_A = \{\partial^\alpha - \partial^\beta : A\alpha = A\beta;\ \alpha, \beta \in \mathbb{N}^d\}. \tag{44}$$

The GKZ hypergeometric system is the left ideal $H(A, u)$ in the Weyl algebra generated by the union of I_A and $A\theta - u$. Solutions of GKZ systams are called A-hypergeometric functions.

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GKZ stands for Gelfand, Kapranov and Zelevinsky, who first studied the general multivariable hypergeometric systems associated to A, u.

Euler-Gauss function as GKZ hypergeometric system

As it has been allready seen before, the multivariable Euler-Gauss function satisfies GKZ system associated to the data

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{45}$$

and $\bar{u} = (-u, -v, 1 - w)$.

Appell function as GKZ hypergeometric system

Consider a GKZ system associated to the following data:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix} \tag{46}$$

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Theese data correspond to the function Φ associated with Appell F_1.

'Hypergeometric properties' of GKZ system

Solutions of GKZ system have properties analogous to the classical (including Euler-Gauss) hypergeometric functions. In "*Generalized Euler integrals and A-hypergeometric functions* (Adv. Math. 84, 255–271), Gelfand, Kapranov and Zelevinsky proved the following

Theorem (GKZ)

Let $f_1, f_2, ..., f_n \in \mathbb{C}[x_1, x_2, ..., x_m]$, $x, \beta \in \mathbb{C}^n$ *and* $\alpha \in \mathbb{C}^n$. *Then*

$$\int_C f_1^{\alpha_1} f_2^{\alpha_2} ... f_n^{\alpha_n} x^\beta dx. \tag{47}$$

where C *is an* m*-dimensional real cycle, are A-hypergeometric functions of the coefficients of the polynomials* $f_1, f_2, ..., f_n$.

The Γ-series and Mellin-Barnes integral

Solutions of GKZ system can be represented as Γ-series

$$\sum_m \prod_{j\in J} \frac{t^{m_j}}{m_j!} \prod_{i\in I} \frac{t^{(Am)_i+u_i}}{\Gamma((Am)_i+u_i+1)}. \tag{48}$$

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There is also a Mellin-Barnes representation, which can be regarded as continuous analog of the Γ-series.

Mordell-Tornheim GKZ-system

The Mordell-Tornheim GKZ-system is associated to the Lie-theoretic interpretation of the series

$$\sum_{m,n\geqslant 0} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_1')_n(\beta_2)_m(\beta_2')_n}{(\eta)_m(\eta')_n(\gamma)_{m+n}\, m!\, n!}\, x^m y^n. \tag{49}$$

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We read the parameter vector
$\boldsymbol{\gamma} = (-\alpha, -\beta_1, -\beta_1', -\beta_2, -\beta_2', \eta-1, \eta'-1, \gamma-1, 0, 0)$ and the B-matrix

$$B = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}. \tag{50}$$

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The A-matrix is computed from the relation $A = B^{\perp}$ and GKZ parameter $\boldsymbol{\alpha} = A\boldsymbol{\gamma}$.

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and the parameter $\boldsymbol{\alpha}$ is equal to $A\boldsymbol{\gamma}$:

$$\boldsymbol{\alpha} = \begin{pmatrix} -1-\alpha+\gamma \\ -1-\beta_1-\beta_2+\eta \\ -1-\beta_2'-\beta_1'+\eta' \\ -\beta_1+\beta_2+\beta_2'-\beta_1' \\ -2+\eta+\eta' \\ -\beta_1+\beta_2-\beta_2'+\beta_1' \\ -\alpha+\beta_2+\beta_1'-\gamma+\eta \\ -\alpha+\beta_2+\beta_1'-\gamma+\eta' \end{pmatrix}. \tag{52}$$

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Now we will deliver analogous formula for $\mathrm{Li}_{\mathrm{MT}}$.

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We have

$$\theta_1^{s_1}\theta_2^{s_2}\theta_3^{s_3}\mathrm{Li}_{\mathrm{MT},s_1,s_2,s_3}(t) = \sum_{m,n>0}(t_1 t_3)^m (t_2 t_3)^n = \frac{t_1^2 t_2 t_3}{(1-t_1 t_3)(1-t_2 t_3)}. \quad (53)$$

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Thus

$$\mathrm{Li}_{\mathrm{MT},s_1,s_2,s_3}(t) = \int_\gamma \frac{dt_1}{t_1}^{\circ s_1-1}\frac{dt_2}{t_2}^{\circ s_2-1}\frac{dt_3}{t_3}^{\circ s_3-1}\circ\frac{dt_1\wedge dt_2\wedge dt_3}{(1-t_1t_3)(1-t_2t_3)}.$$

Generating function and associated differential equation

The Drinfeld-Kontsevich integral can be used to construct Fuchsian differential equation associaded to generating function of the sequence $\zeta(\{(s_1, s_2, ..., s_p)\}^n)$.

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First, note that the generating function $F(\lambda, 1)$ of (multiple) zeta vealues satisfies

$$\begin{aligned} F(\lambda, t) &= 1 - \lambda^s \mathrm{Li}_s(t) + \lambda^{2s} \mathrm{Li}_{s,s}(t) - ... \\ &= 1 - \lambda^s T_s^{-1}.1 + \lambda^{2s} T_s^{-2} 1 - ... \\ &= [1 + \lambda^s T_s^{-1}]^{-1}.1. \end{aligned} \tag{54}$$

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Thus $F(\lambda, t)$ is a solution of the eigen-equation of T_s.

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We define opertor $T = T_{s_1,s_2,...,s_p}$ as:

$$T := (1-t)\partial_t (t\partial_t)^{s_1-1} ... (1-t)\partial_t (t\partial_t)^{s_p-1}. \tag{56}$$

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Holomorphic solution $F(t,\lambda)$ of the eigenequation

$$(T+\lambda^{|s|})f = 0, \tag{57}$$

such that $F(1,0)=1$, has the following expansion around $t=1$:

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In other words, function $F(1,\lambda)$ is a generating function of the sequence $\zeta(\{s_1,\ldots,s_p\}^n)$.

Particular solutions associated to $\zeta(\{s\}^n)$

If the depth p is equal to one, then T has the form

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We have

$$F(t,\lambda) \;=\; {}_sF_{s-1}\left(\left.\begin{matrix}\mu\lambda,\mu^2\lambda,...,\mu^s\lambda\\ 1,..,1\end{matrix}\;\right|\;t\right). \tag{61}$$

M.-T. generating function

In a similar way, as in the case of classical polylogarithm, knowing that the PDE system $T_{\mathrm{MT},s_1,s_2,s_3}$ applied to the period integral associated to $\zeta_{\mathrm{MT},s_1,s_2,s_3}$ gives 1, we get

$$\left[T_{\mathrm{MT},1,1,1} + \lambda^3\right].\Phi \;=\; 0. \tag{62}$$

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Suitably choosen solution $\Phi(\lambda, t_1, t_2, t_3)$ is the generating function of Mordell-Tornheim polylogarithms (and multiple zeta-values, after putting $x = y = 1 = t_1 = t_2 = t_3$). It is a GKZ-hypergeometric function.

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General case

In general, we have the following.

Problem

Is there a way to associate (generalized) hypergeometric function to every (arithmetic) Shintani zeta-function?

THANK YOU!

- Gelfand I., *General theory of hypergeometric functions*. Soviet Math. Dokl. 33 (1986), p. 573 - 577.
- Gelfand I., Gelfand S., *General theory of hypergeometric systems*. Soviet Math. Dokl. 33 (1986), p. 279 - 283.
- Gelfand I., Graev M., *GG Functions and their Relations to General Hypergeometric Functions*. Letters in Math. Phys. 50 no. 1 (1999), p. 1 - 28.
- Gelfand I., Kapranov M., Zalevinsky A. V., *Equations of hypergeometric type and Newton polytopes*. Soviet Math. Dokl., 37 (1988), p. 678 - 683.
- Gelfand I., Kapranov M., Zalevinsky A. V., *Hypergeometric functions and toric varieties*. Funct. Anal. and Appl. 23 no. 2 (1989), 12 - 26.
- Graev M., *General Hypergeometric Functions*. Funct. Anal. and Appl. 26 no. 2 (1992), 131 - 133.
- Kapranov M., *Hypergeometric functions on reductive groups*. In "Integrable systems and algebraic geometry" (M.H. Saito, Y. Shimizu, K. Yeno Eds.), p. 236 - 281, World Scientific Publ. 1998.

Stienstra J., *GKZ hypergeometric structures*. math.AG/0511351.

Zagier D., *Values of zeta functions and their applications*. in First European Congress of Mathematics (Paris, 1992), Vol. II, A. Joseph et. al. (eds.), Birkhäuser, Basel, 1994, str. 497-512

Żołądek H. *The monodromy group*. Birkhäuser, Basel, 2006.