The center problem for polynomial Abel equation

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The Abel quation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}P}{\mathrm{d}x}y^2 + \frac{\mathrm{d}Q}{\mathrm{d}x}y^3,$$

where $P(x)$ and $Q(x)$ are polynomials with complex coefficients such that $P(0) = Q(0) = 0$.

For $y_0 \in \mathbb{C}$ sufficiently close to 0 the initial value problem $y(0) = y_0$ has unique solution $y = \varphi(x; y_0)$ and the **Poincaré map**

$$y_0 \longmapsto \mathcal{P}(y_0) = \varphi(1; y_0)$$

is well defined. We say that we have **center at** $y = 0$ if

$$\mathcal{P} \equiv id;$$

we call it as the **center condition**. (The polynomials P and Q can be changed by adding constants, so often the Abel equation is written using $p(x) = P'(x)$ and $q(s) = Q'(x)$; also sometimes the boundary points $x = 0$ and $x = 1$ are taken as $x = a$ and $x = b$ respectively.)

M. Briskin, J.-P. Françoise and Y. Yomdin formulated the following **Center Problem** for the Abel equation:

Describe the subvariety of the space of Abel equations consisting of equations which have center at $y = 0$.

(This is a natural generalization of the center problem for polynomial plane vector fields).

They formulated the so-called **Composition Conjecture**:

Each center case corresponds to a situation of the following type:

$$P = \tilde{P} \circ R, \quad Q = \tilde{Q} \circ R, \quad R(0) = R(1) = 0, \quad \deg R > 0,$$

i.e. the polynomials P and Q are compositions with a nonconstant polynomial $R(x)$ which takes the same value at $x = 0$ and at $x = 1$.

That the composition implies the center condition follows from the fact that then Abel equation can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}w} = \frac{\mathrm{d}\tilde{P}}{\mathrm{d}w} y^2 + \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}w} y^3,$$

i.e., Abel equation is a **pull-back** of the latter equation by means of the non-invertible map

$$(x, y) \longmapsto (w, y) = (R(x), y).$$

The aim of my talk is to present a comprehensive approach to the Composition Conjecture. Our method is new, its principal ingredients are analytic on one side and qualitative on the other side. The analytic part involves very precise expansions of some functions (usually first integrals). The qualitative part is mainly topological; we analyze topology of some complex foliations in complex surfaces.

The periodic case and relation with planar vector fields

Consider, for instance, the Abel equation related with a quadratic vector field

$$\dot{\zeta} = i\zeta + A\zeta^2 + B\zeta\bar{\zeta} + C\zeta^2, \quad \zeta \in \mathbb{C} \simeq \mathbb{R}^2.$$

Firstly, we pass to the coordinates (r, x) such that $\zeta = re^{i\varphi} = rx$ and $\bar{\zeta} = rx^{-1}$; we get the equation

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{M(x)r^2}{2i + N(x)r},$$

where $M(x)$ and $N(x)$ are Laurent polynomials.

Next, the variable $y = r/(2i + N(x)r)$ satisfies the Abel equation with $P'(x) = M(x) - N'(x)$ and $Q'(x) = -M(x)N(x)$. For $x = e^{i\varphi}$, i.e., along the circle $|x| = 1$, we get a periodic Abel equation and the center conditions means a center for the palar vector field.

On the other hand, in the case of the Lotka–Volterra type center $Q_3^{LV} = \{B = 0\}$ we know that the quadratic system has first integral of the form $L_1^a L_2^b L_3^c$, where $L_j(\zeta, \bar{\zeta})$ are linear affine functions in generic position and the exponents a, b and c can be arbitrary. Since such first integral is not a composition with a Laurent polynomial $R(x)$, the composition does not take place. (But in the case of reversible quadratic center Q_3^R with real A, B, C (after a suitable rotation of the real phase plane) we have a composition with $R(x) = x - x^{-1}$.)

The Poincaré–Lyapunov quantities

Of course, the **center variety**, i.e., the subvariety of the space of the Abel equations with fixed degrees of P and Q defined by the center condition, is defined by vanishing of the coefficients $c_2, c_3, \ldots$ in the Taylor expansion

$$\mathcal{P}(y_0) = y_0 + c_2 y_0^2 + c_3 y_0^3 + \ldots .$$

By analogy with the classical center–focus problem we call these coefficients the **Poincaré–Lyapunov quantities**. We have

$$\begin{aligned} c_2 &= P(1), \\ c_3 &= P^2(1) + Q(1), \\ c_4 &= P^3(1) + 2P(1)Q(1) + \int_0^1 P \mathrm{d}Q, \end{aligned}$$

and general c_j is a polynomial in coefficients of the polynomials P and Q. Thus the center variety is an algebraic variety. Of course, the conditions

$$P(1) = Q(1) = 0$$

are necessary for the center.

The Moment problem

Note the integrals $\int P\mathrm{d}Q$ and $\int P^2\mathrm{d}Q$ in the formulas for c_4 and c_5. They are the **moments**

$$m_k(P, \mathrm{d}Q) = \int_0^1 P^k \mathrm{d}Q$$

of the variable P with respect to the (complex) measure $\mathrm{d}Q$ on $[0,1]$. Also further quantities c_j are expressed via these moments. However, there appear more complicated integrals, like $\int P^2 Q\mathrm{d}Q$ in c_7.

The moments m_k play essential role in the so-called weakened center problem, or the **Moment Problem**. Consider the situation when the polynomial $Q(x)$ is small in the sense that

$$Q(x) = \varepsilon Q_0(x),$$

where ε is a small parameter. Then the variation analysis leads to $y(x;y_0;\varepsilon) = \varphi_0(x;y_0) + \varepsilon\varphi_1(x;y_0) + \ldots$ where $\varphi_0(x;y_0) = y_0/(1-P(x)y_0)$,

$$\varepsilon\varphi_1(x;y_0) = \frac{y_0^3}{(1-P(x)y_0)^2}\int_0^x \frac{\mathrm{d}Q(s)}{1-P(s)y_0}$$

and the center condition $\varphi_1(1;y_0) = 0$ implies $I(c) \equiv 0$ where

$$I(c) = \int_0^1 \frac{\mathrm{d}Q(x)}{P(x)-c} = -\frac{1}{c}\sum_{k=0}^{\infty} m_k(P,\mathrm{d}Q)\cdot c^{-k}$$

(and $c = 1/y_0$).

For some time there existed a conjecture that vanishing of all the moments $m_k(P,\mathrm{d}Q)$ implies the composition. Colin Christopher proved this under the assumption that $P'(0)P'(1) \neq 0$.

But F. Pakovich constructed the following example:

$$P(x) = T_6(x) \quad Q(x) = T_2(x) + T_3(x),$$

where $T_n(x)$ are the Chebyshev polynomials defined by $T_n(\cos t) = \cos nt$ (e.g., $T_2 = 2x^2 - 1$, $T_3 = 4x^3 - 3x$).

Here the Poincaré map is $\mathcal{P} : y(-a) \longmapsto y(a)$, where $a = \sqrt{3}/2$ is the positive zero of T_3; thus $T_{2,3}(a) = T_{2,3}(-a)$. All the moments $m_k(P, dQ) = \int_{-a}^{a} T_6^k dT_2 + \int_{-a}^{a} T_6^k dT_3$ vanish, because $T_6 = T_2 \circ T_3 = T_3 \circ T_2$. Calculations give $c_5 \neq 0$ which demonstrates that there is no center at $y = 0$ and that there is no composition. Note also that $P'(\pm a) = 4T_3(\pm a)T_3'(\pm a) = 0$, so the assumption of the Christopher's theorem fails.

F. Pakovich with M. Muzychuk proved that $m_k(P, \mathrm{d}Q) = 0$, $k = 0, 1, 2, \ldots$, implies that $Q(x) = \tilde{Q}_1 \circ R_1(x) + \ldots + \tilde{Q}_r \circ R_r(x)$, where

R_j are different composition 'factors' of P, i.e., $P = \tilde{P}_j \circ R_j$ and $R_j(0) = R_j(1)$.

Some progress in the Center Problem was obtained by Briskin Roytvarf and Yomdin, where a suitable projectivisation of the space of pairs of polynomials (P, Q) with given degree was introduced. Values of the Poincaré–Lyapunov quantities at infinity (in the projective space) are reduced to ordinary moments. This implies (under some restrictions onto degrees which follow from positive solutions of the composition conjecture for the Moment Problem) that any component of the center variety of positive dimension (i.e., with moduli) obeys the Composition Conjecture. There remain eventual isolated pairs (P, Q) to be considered.

The Liénard equation and the Lüroth theorem

Let us present an approach to our problem using the **Liénard equation**

$$z\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{\mathrm{d}Q}{\mathrm{d}x} - \frac{\mathrm{d}P}{\mathrm{d}x}z.$$

This equation arises from the Abel equation by the substitution

$$z = 1/y.$$

It is considered near the **line at infinity**

$$\mathbb{E}_\infty = \{z = 0\} = \{y = \infty\}$$

and has **singular points**

$$M_j = \mathbb{E}_\infty \cap \{x = X_j\} = \big(X_j, \infty\big)$$

such that $Q'(X_j) = 0$, $j = 1, \ldots, s$; due to the condition $Q(0) = Q(1) = 0$ at least one such point does exist.

Suppose that $Q(x) = a(x - X_0)^2 + \ldots$, $a > 0$, near X_0. Thus M_0 is of the center or focus type. L. Cherkas proved that M_0 is a center if and only if the polynomial map

$$\Delta : x \longmapsto (X, Y) = (P(x), Q(x)),$$

is a multiple covering from $\mathbb{C}$ to the algebraic curve

$$C := \Delta(\mathbb{C}).$$

By the **Lüroth theorem** this implies existence of a composition $P = \tilde{P} \circ R$, $Q = \tilde{Q} \circ R$ such that $R'(X_0) = 0$.

The **Lüroth theorem.** *If $\Delta : M \longmapsto N$ is a ramified covering between Riemann surfaces such that $M = \mathbb{CP}^1$ then also $N = \mathbb{CP}^1$.*

It easily follows from the Riemann–Hurwitz formula.

In our situation we have $M = \mathbb{C} \cup \infty$, $N = C \cup \infty$ and $\Delta = (P, Q)$ has the property $\Delta^{-1}(\infty) = \infty$. The polynomial R appears in the diagram $M \overset{R}{\longmapsto} \mathbb{CP}^1 \overset{I}{\longmapsto} N$, where I is the isomorphism between $\mathbb{CP}^1$ and N , $I(\infty) = \infty$, and $\Delta = R \circ I$.

The same is true in the saddle case, $a < 0$, but here the center condition is replaced with the integrability condition, i.e., the existence of a local analytic first integral. For example, if the polynomials P and Q are real and $Q = a\left[(x - 1/2)^2 - 1/4\right]$, $a < 0$, then looking at the real phase portrait of the Liénard equation we see that the Dulac map between separatrices of the saddle $M_1 = (1/2, 0)$ must be analytic. So the saddle is integrable and

we have a composition with $R = Q$ (by the Cherkas' argument).
This proof is due to C. Christopher.

The Abel foliation

The Abel equation defines a holomorphic foliation $\mathcal{F}_0$ in the open complex surface $\mathbb{C} \times \mathbb{C}$. Its leaves near the line $\mathbb{E}_0 = \{y = 0\}$ are represented as graphs of solutions to the Abel equation,

$$\mathcal{L} = \mathcal{L}(y_0) = \{(x, \varphi(x; y_0) : x \in \mathcal{D}(\varphi) \subset \mathbb{C}\}.$$

Above $\mathcal{D}$ is the domain of definition of the solution $\varphi(\cdot; y_0) = \varphi_{\mathcal{L}(y_0)}$.

The center condition means existence of a path $\gamma_{\mathcal{L}} \subset \mathcal{L}$ which joins the points $(0, y_0) \in \mathbb{D}_0$ and $(1, y_0) \in \mathbb{D}_1$. Here

$$\mathbb{D}_x = \{x = \text{const}\}.$$

For small $|y_0|$ the path $\gamma_{\mathcal{L}}$ is a lift to $\mathcal{L}$ of the straight segment $\delta_{\mathcal{L}} = [0,1]$ in the $x-$plane, $\delta_{\mathcal{L}} = \pi(\gamma_{\mathcal{L}})$. This lift uses the **vertical projection**

$$\pi : (x, y) \longmapsto x$$

with the fibers $\mathbb{D}_x$.

The domain $\mathcal{D}\,(\varphi)$ of definition of the function φ is not the whole $x-$plane, because the solutions usually escape to infinity. For small initial condition y_0 the domain $\mathcal{D}_{\mathcal{L}(y_0)}$ is large, but for large $|y_0|$ the construction of the path $\gamma_{\mathcal{L}}$ is not obvious.

For large $|y_0|$ it is natural to consider foliation by integral curves of the Liénard equation near the line $\mathbb{E}_\infty = \{y = 1/z = \infty\}$. In this way we obtain a holomorphic foliation in $\mathbb{C} \times \mathbb{CP}^1$ which is a natural prolongation of the foliation $\mathcal{F}_0$.

We can define a foliation of $\mathbb{CP}^1 \times \mathbb{CP}^1$ called the **Abel foliation** and denoted by $\mathcal{F}$.

Lemma *The foliation $\mathcal{F}$ is singular and its singularities are isolated and coincide with the singular points M_j of the Liénard equation and possibly with some singular points at the line $\mathbb{D}_\infty = \{x = \infty\}$.*

Lemma *Near the line $\mathbb{E}_\infty$ and away from the singular points M_j the leaves $\mathcal{L}$ of the foliation $\mathcal{F}$ meet $\mathbb{E}_\infty$ vertically:*

$$\mathcal{L} = \left\{x - x^{\mathcal{L}} = a^{\mathcal{L}} z^2 + \ldots\right\},$$

where $a^{\mathcal{L}} \neq 0$ if

$$q^{\mathcal{L}} = (x^{\mathcal{L}}, \infty) \in \mathcal{L} \cap \mathbb{E}_\infty$$

is not a singular point of $\mathcal{F}$.

The latter lemma implies that typical singularities of the solutions $\varphi(x)$ are ramification points. In fact, a typical solution (of a typical Liénard equation) has infinitely many such ramification points. Presence of these points causes that the Riemann surface of the multivalued holomorphic function φ is complicated.

Also the path $\gamma_{\mathcal{L}}$, which is quite simple for $\mathcal{L} = \mathcal{L}(y_0)$ with small $|y_0|$, must be deformed when $|y_0|$ grows. It must avoid the points $q^{\mathcal{L}}$ at infinity and $\delta_{\mathcal{L}}$ must avoid the ramification points $x^{\mathcal{L}} = \pi(q^{\mathcal{L}})$; moreover, this avoiding must occur in a precise way (in order to be at a correct branch of the Riemann surface).

The Abel foliation has natural first integral $\boldsymbol{F}_0$ near the line

$$\mathbb{E}_0 = \{y = 0\}\,.$$

By definition we have

$$\boldsymbol{F}_0(x, y) := y_0,$$

where y_0 is the initial value of the solution $y = \varphi(x; y_0)$ (or a leaf $\mathcal{L}(y_0)$) such that $\varphi(0; y_0) = 0$. We call it the **first integral at zero**. The center condition is interpreted as follows:

$$\boldsymbol{F}_0(0, y) \equiv \boldsymbol{F}_0(1, y).$$

Of course, $\boldsymbol{F}_0$ is well defined and single valued in some domain $\mathcal{U}(\boldsymbol{F}_0)$.

The first integral at infinity

The Liénard equation near $\mathbb{E}_\infty$ gives an equation for the phase curves in the Pfaff form

$$\mathrm{d}Q(x) + z\mathrm{d}z + z\mathrm{d}P(x) = 0.$$

The first two terms in the above Pfaff form define a complete differential $\mathrm{d}\Big(Q + z^2/2\Big)$ and the whole Pfaff equation can be regarded as a perturbation of it. Let us then assume first integral in form of a power series in z :

$$\boldsymbol{F}_\infty(x,z) = f_0(x) + f_1(x)z + f_2(x)z^2 + \ldots.$$

We arrive at the following system for the coefficient functions:

$$\begin{array}{rcl} f_1 Q' & = & 0, \\ 2f_2 Q' + P' f_1 & = & f_0', \\ 3f_3 Q' + 2P' f_2 & = & f_1', \\ \dots\dots\dots\dots\dots\dots\dots\dots & \dots & \dots \\ n f_n Q' + (n-1) f_{n-1} P' & = & f_{n-2}'. \end{array}$$

The first two equations are solved in the expected form:

$$f_1 = 0, \quad f_0 = Q, \quad f_2 = 1/2.$$

Then the further equations allow to define other coefficients in an algebraic way,

$$f_n = \frac{1-n}{n} \frac{\mathrm{d}P}{\mathrm{d}Q} f_{n-1} + \frac{1}{n} \frac{\mathrm{d}f_{n-2}}{\mathrm{d}Q}, \quad n \geq 3,$$

where we have replaced $F'/q = F'/Q'$ with $\mathrm{d}F/\mathrm{d}Q$. Since $q(x) = Q'(x)$ and $p(x) = P'(x)$ are polynomials we find that the coefficients $f_n(x)$ are rational functions with poles at the zeroes X_j of the polynomial $q(x)$, i.e., at the critical points of $Q(x)$ which correspond to singular points $M_j = (X_j, 0)$ of the Liénard equation. In particular, we have $f_3 = -(1/3)\cdot \mathrm{d}P/\mathrm{d}Q$, $f_4 = (1/4)\cdot(\mathrm{d}P/\mathrm{d}Q)^2$, $f_5 = -(1/15)\cdot \mathrm{d}^2P/\mathrm{d}Q^2 - (1/5)\cdot(\mathrm{d}P/\mathrm{d}Q)^3$.

$f_j,\ j \geq 3$, are polynomials in $\mathrm{d}^iP/\mathrm{d}Q^i$ such that f_{2k} depends on $\mathrm{d}^iP/\mathrm{d}Q^i$ *for* $i \leq k-1$ and

$$f_{2k+1} = \alpha_k \frac{\mathrm{d}^kP}{\mathrm{d}Q^k} + \left(\text{terms depend. on } \frac{\mathrm{d}P^i}{\mathrm{d}Q^i},\ i < k\right), \quad \alpha_k \neq 0.$$

The first integral $\boldsymbol{F}_\infty$ is called the **first integral at infinity**.

Note also that this integral is defined by the ‘boundary condition’

$$\boldsymbol{F}_\infty|_{\mathbb{E}_\infty} = Q(x).$$

We claim that the series defining $\boldsymbol{F}_\infty$ is convergent in some domain. Indeed, since $\boldsymbol{F}_\infty|_{\mathbb{E}_\infty}$ is a regular function in $\mathbb{E}_\infty \setminus \{M_1, \ldots, M_s\}$, the first integral $\boldsymbol{F}_\infty$ is also regular in the domain where the phase curves of the Liénard vector field do not accumulate. The only possible accumulation points of these phase curves are singular points and their separatrices, including separatrices of eventual singular point M_∞ at $\mathbb{D}_\infty = \{x = \infty\}$.

The domain of definition of the first integral at infinity can be enlarged. We define **local cut sets** Σ_j which are CW complexes. Each such Σ_j is a union of finite number of strata of different dimension. It consists of: 3–dimensional cells (whose boundaries lie either on separatrices of the point M_j or on other local cut sets Σ_i) or 4–dimensional strata (called black boxes). Their union is called the **cut set**.

When taking intersections of Σ_j with the horizontal lines $\mathbb{E}_y = \{(x, y) : x \in \mathbb{C}\}$, then we obtain a collection of arcs (often segments) and discs (corresponding to the black boxes) $\Sigma_j(y) = \Sigma_j \cap \mathbb{E}_y$. When $|y|$ is large (i.e., $|z| = 1/|y|$ is small) then $\Sigma_j(y)$ is localized near the points (X_j, y) (corresponding to the singular points M_j at infinity).

When $|y|$ decreases the arcs and discs in $\Sigma_j(y)$ grow. At some moment the local sections $\Sigma_j(y)$ and $\Sigma_i(y)$ associated with different singular points become intersecting. Therefore some of the arcs become shortened, e.g. when an endpoint of one arc $\Sigma_i(y_0)$ meets another arc $\Sigma_j(y_0)$ at interior point then for larger $|y|$ the arc $\Sigma_i(y)$ will end at a point in $\Sigma_j(y)$. So one obtains a kind of graph. For $|y|$ small enough the graph $\bigcup \Sigma_j(y)$ becomes connected and divides the plane $\mathbb{E}_y$ into finite number of separated domains.

Of course, the local cut set is not uniquely defined. Its strata Σ_j can be deformed and we will do this.

The Christopher condition

The points $(x, z) = (0, 0) = \mathbb{D}_0 \cap \mathbb{E}_\infty$ *and* $(1, 0) = \mathbb{D}_1 \cap \mathbb{E}_\infty$ *are not singular for the foliation* $\mathcal{F}$*, i.e.*

$$Q'(0)Q'(1) \neq 0.$$

Proposition *Under the above assumption the center condition is equivalent to the condition*

$$\boldsymbol{F}_\infty(0, z) \equiv \boldsymbol{F}_\infty(1, z)$$

for small $|z|$.

'Proof'. We know two first integrals for the Abel foliation. One is $\boldsymbol{F}_\infty$, the first integral at infinity. The other is the first integral at zero $\boldsymbol{F}_0$. Both are defined in different regions of the phase space $\mathbb{C} \times \mathbb{CP}^1$, $\mathcal{U}(\boldsymbol{F}_\infty)$ near $\mathbb{E}_\infty \setminus \{M_1, \ldots, M_s\}$ and $\mathcal{U}(\boldsymbol{F}_0)$ near $\mathbb{E}_0$.

The center condition means the identity

$$\boldsymbol{F}_0(0, y) \equiv \boldsymbol{F}_0(1, y).$$

If the domains of definition of $\boldsymbol{F}_\infty$ and of $\boldsymbol{F}_0$ intersected, then second identity would imply the first identity.

We choose a path

$$\theta = \{(0, y_0(s)) : 0 \leq s \leq 1\}$$

in the plane $\mathbb{D}_0$ which begins at $(0, 0)$, i.e., $y_0(s) = 0$ and such that its endpoint $(0, y_0(1))$ lies in the domain of analyticity of the function $\boldsymbol{F}_\infty$. We have a family $\{\mathcal{L}(s)\}_{0\le s\le 1}$, $\mathcal{L}(s) = \{\mathcal{L}(y_0(s))$, of leaves of the Abel foliation. We construct a family $\left\{\gamma_{\mathcal{L}(s)}\right\}$ of paths on the leaves $\mathcal{L}(s)$. In this construction we care that each $\gamma_{\mathcal{L}(s)}$ is always finite (lies in $\mathbb{C}\times\mathbb{C}$) and hence its projection $\delta_{\mathcal{L}(s)}$ avoids the ramification points $x_j^{\mathcal{L}(s)} = \pi(q_j^{\mathcal{L}(s)})$ in correct way. Moreover, the final path $\gamma_{\mathcal{L}(1)}$ should lie completely in the domain of analyticity of $\boldsymbol{F}_\infty$, i.e., it should avoid the cut set Σ. For this aim one deforms the cut set; it needs some work, but can be done.

If such construction can be done than the first integral $\boldsymbol{F}_0$ can be prolonged to some domain which contains the (real) surface $\Gamma = \bigcup_s \gamma_{\mathcal{L}(s)}$. This domain meets the domain of definition of the first integral $\boldsymbol{F}_\infty$ and we are done.

Recall that $Q(0) = Q(1) = P(0) = P(1) = 0$. Let

$$X = \sum_{j \geq 1} a_j^{(0,1)} Y^j$$

be the Taylor expansions of the germ at $x = 0, 1$ of the curve $\Delta : x \mapsto (X, Y) = (P(x), Q(x))$. Then the restrictions of the first integral at infinity to the vertical lines $\mathbb{D}_i$, $i = 0, 1$, have the Taylor expansions

$$\boldsymbol{F}_\infty(i, z) = \sum_{j \geq 1} c_j(a^{(i)}) z^j,$$

such that

$$c_{2k+1} = \beta_k a_k^{(i)} + \tilde{c}_{2k+1}(a_1^{(i)}, \ldots, a_{k-1}^{(i)})$$

$$c_{2k} = c_{2k}(a_1^{(i)}, \ldots, a_{k-1}^{(i)}), \;\; \beta_k \neq 0,$$

where c_j are universal polynomials of $a = (a_1, a_2, \ldots)$.

The condition $\boldsymbol{F}_\infty(0, z) \equiv \boldsymbol{F}_\infty(1, z)$ implies that the two local curves

$$(\mathbb{C}, i)\, x \longmapsto \Delta(x) \in (C, (0,0)),$$

$i = 0, 1$, where $\Delta(x) = (P(x), Q(x))$, are identical.

Therefore the parametric curve $\mathbb{C} \xmapsto{\Delta} C$ is a multiple covering and application of the Lüroth theorem proves the Composition Conjecture under the Christopher condition.

The Bogdanov-Takens foliation

Now we assume that the Christopher condition fails, i.e., at least one of the points $(0,\infty)$ or $(1,\infty)$, is singular, equal $M_j = (X_j,\infty)$. This is the Bogdanov-Takens singularity

$$\dot{x} = z, \quad \dot{z} = -B'(x) - A'(x)z,$$

where

$$A(x) = P(X_j + x) = a_m x^m + a_{m+1} x^{m+1} + \dots a x^m + \dots$$

and $B(x) = Q(X_j + x) = bx^n + \dots$.

Moreover, using the change

$$x \longmapsto B^{1/n}$$

we can assume

$$B(x) = x^n$$

Note that the latter change is invertible and compatible with the Composition Conjecture. It is a local pull back.

Further analysis depends on the multiplicities m and n. We divide the set of germs into three classes:

- of the **cusp type** if $m > n/2$;
- of the **general type** if $n = 2m$;

- of the **saddle–node type** if $m < n/2$.

We focus our attention on the general type; other types are simpler and we omit them.

The general type

Recall that we deal with the following vector field

$$\begin{aligned}
V &= V_0 + V_1, \\
V_0 &= z\partial_x - mx^{m-1}(2x^m + az)\partial_z, \\
V_1 &= -\{(m+1)a_{m+1}x^m + \ldots\}z\partial_z,
\end{aligned}$$

where $a = a_m \neq 0$. The above splitting is clear when one introduces the following **quasi-homogeneous grading**:

$$\deg_H x = 1, \qquad \deg_H z = m.$$

Then the vector field V_0 is quasi-homogeneous of $\deg_H = m-1$ and V_1 contains terms of greater degree.

Lemma *The vector field V_0 is a pull-back, by means of the mapping*

$$x \longmapsto \tilde{x} = x^m$$

(ramified for $m > 1$) with subsequent division by mx^{m-1}, of the linear vector field

$$W_0 = z\partial_{\tilde{x}} - (2\tilde{x} + az)\partial_z.$$

Therefore vector field V_0 has the Darboux first integral

$$\begin{aligned} \boldsymbol{H}_0 &= (x^m - z/\lambda_1)^{\mu_1}(x^m - z/\lambda_2)^{\mu_2}, \quad a^2 \neq 8, \\ \boldsymbol{H}_0 &= \left(x^m + z/\sqrt{2}\right)^2 \exp\left(\frac{-\sqrt{2}z}{x^m + z/\sqrt{2}}\right), \quad a = 2\sqrt{2}. \end{aligned}$$

Above the quantities

$$\lambda_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 8}\right), \quad \mu_{1,2} = 1 \mp \frac{a}{\sqrt{a^2 - 8}}$$

are related with the eigenvalues $\lambda_{1,2}$ of the linear vector field W_0; the characteristic equation is $\lambda^2 + a\lambda + 2 = 0$.

The case $a^2 = 8$ is called the $1:1$ **resonant case**, because the linear vector field Y_0 has $2-$dimensional Jordan cell; here we can assume $a = 2\sqrt{2}$ because of the possible change $x \longmapsto e^{2\pi i/m}x$. The integer m is called the **ramification index**.

We make the following additional division. We say that such germ is **saddle–like** if

$$\text{Re}\mu_1 > 0 \text{ and } \text{Re}\mu_2 > 0$$

and it is **node–like** if

$$\text{Re}\mu_1 \cdot \text{Re}\mu_2 \leq 0.$$

Since $\mu_1 + \mu_2 = 2$ this classification is well defined.

Of course, the curves

$$S_{1,2} = \left\{ z = \lambda_{1,2} x^m \right\}$$

are separatrices of the singular point $x = z = 0$; for $a = 2\sqrt{2}$ there is only one separatrix.

Similar situation (with some subtlietes) holds for the whole vector field V, i.e., the separatrices are approximately as above.

Let us concentrate on the case $m = 1$.

The **saddle–like case.** Let

$$x = \psi_j(z) = e_j z + \dots, \quad j = 1, 2,$$

denote denote the branches of the function x restricted to the separatrices $S_1 \cup S_2$. Then the **local cut set** equals

$$\Sigma_{\text{loc}} = \bigcup_z \{(s\psi_1(z), z) : 0 \leq s \leq 1\} \cup \{(s\psi_2(z), z) : 0 \leq s \leq 1\}$$

(where $|z| < \epsilon$). It means that the intersection $\Sigma_{\text{loc}}(1/z)$ of Σ_{loc} with a fixed plane $\{z = \text{const}\}$ is a segment (star for $m > 1$) which joins the center $x = 0$ with the outer vertices $\Psi_j(z) = (\psi_j(z), z)$. The intersection of Σ_{loc} with a fixed plane $\{x = \text{const}\}$ is a union of arcs which begin at the points $z = \lambda_{1,2}x^m + \ldots$ and go in the direction opposite the origin.

In the case when the singular point is critical for the original center problem, i.e., it belongs to one of the end lines $\mathbb{D}_0 = \{x = 0\}$ or $\mathbb{D}_1$, we modify the definition of the local cut set to the following **modified local cut set**:

$$\Sigma_{\mathsf{mloc}} = \bigcup_{j=1,2} \bigcup_{z} \left\{ \left(s\psi_j(z) + (1-s)\epsilon z^{1/m}, z \right) : 0 \le s \le 1 \right\}.$$

In the **node–like case different from the** $1:1$ **resonant node** the **local cut set**, respectively, the **modified local cut set**, consists of one separatrix S_j (corresponding to Re$\mu_j > 0$) and a **black box** Ω suitably connected; either by a family stars with center at $x = 0$ at the complex lines $\mathbb{E}_{1/z}$ or by stars with center slightly moved from zero (analogously like in the saddle–like case). (Inside the black box the integral $\boldsymbol{H}_0$ is big and the situation is out of control.)

In the $1:1$ **resonant and unramified (**$m = 1$) **case** the local cut set Σ_{loc} coincides with the black box Ω. If $m > 1$ then Σ_{loc}

is constructed like above: with the black boxes connected by a family of stars in the planes $\mathbb{E}_{1/z}$.

The differential system rewritten in the resolution coordinates

$$x, u = z/x^m$$

(and divided by x^{m-1}) takes the form

$$\dot{x} = ux, \quad \dot{u} = -m(u^2+au+2)-u\Big\{(m+1)a_{m+1}x + (m+2)a_{m+2}x^2 + \ldots$$

We look for local first integral in form of a series

$$\boldsymbol{H} = h_{2m}(u)x^{2m} + h_{2m+1}(u)x^{2m+1} + \ldots,$$

where $h_{2m}(u)x^{2m} = \boldsymbol{H}_0$. Then the coefficient functions $h_d(u)$, $d > 2m$ satisfy the recurrent equations

$$m\left(u^2 + au + 2\right) h_d' - duh_d = f_d,$$
$$f_d(u) = -(d-m)a_{d-m}uh_{2m}'(u) + \tilde{f}_d(u),$$

where $\tilde{f}_d(u)$ depend on the coefficients $a_{m+1}, \ldots, a_{d-m-1}$. The solutions are chosen in the form

$$h_d(u) = \frac{1}{m} h_0^{d/2m} \int_0^u h_0(v)^{-d/2m} \frac{f_d(v)}{v^2 + av + 2} \mathrm{d}v,$$

i.e., such that $h_d(0) = 0$ and hence $\boldsymbol{H}(x, 0) = x^{2m}$.

We are interested in the prolongation of the series defining $\boldsymbol{H}$ to the line $\{x = 0\}$. We encounter the question of the choice of the integration path in the above formula, the subintegral function has singularities at $v = \lambda_1$ and $v = \lambda_2$. Even the prolongation of the first term $\boldsymbol{H}_0 = h_{2m}(u)x^{2m}$ is not obvious at all. This problem is directly related with the choice of the modified local cut set.

Consider the case when the singular points are not real, $\mathrm{Im}\lambda_2 < 0 < \mathrm{Im}\lambda_1$; this takes place for generic parameter a. Then we can choose the modified local cut set in two ways.

In one case we take the continuation of $\boldsymbol{H}(x,z)$ corresponding to movement along the right real half-line $L_+ = \{u \geq 0\}$, from $u = 0$ to $u = +\infty$, and the modified local cut set lies on the left of L_+. This corresponds to choice of a path $\vartheta_{\mathcal{L}} \subset \mathcal{L}(1/z)$ from $(0,z)$ to $(x,0)$. We get

$$H_0(x,z)|_{x=0} = c^+(a)z^2,$$

where

$$c^+(a) = \lambda_1^{-\mu_1}\lambda_2^{-\mu_2}.$$

But, when we approach $u = \infty$ along the left half-line $L_- = \{u \leq 0\}$ (with the local modified cut set on right of it), then we get

$$H_0(x,z)|_{x=0} = c^-(a)z^2$$

where

$$c^-(a) = c^+(a) \cdot e^{-2\pi \mathrm{i}\mu_1} = c^+(a) \cdot e^{2\pi \mathrm{i}\mu_2},$$

because this change is the result of the monodromy of the multivalued function $h_{2m}(u)$ along a loop in $\widehat{\mathbb{C}}$ with the base point $u = \infty$ and turning around $u = \lambda_1$ in negative direction. (Recall that $\mu_1 + \mu_2 = 2$).

The continuations of the further terms $h_d(u)x^d$, $d = m + k$, to $\{x = 0\}$ lead to $c_k^{\pm} z^{1+k/m}$ with

$$\begin{aligned} c_k^{\pm}(a) &= \Big(c^{\pm}(a)\Big)^{d/2m} \int_{L_\pm} h_{2m}(u)^{-d/2m} \frac{f_d(u)\mathrm{d}u}{u^2 + au + 2} \\ &= \Big(c^{\pm}(a)\Big)^{d/2m} \cdot \frac{m-d}{2} \cdot J_k^{\pm}(a) \cdot a_{d-m} + \tilde{c}_k^{\pm}, \end{aligned}$$

where $\tilde{c}_k^{\pm}(a, a_{m+1}, \ldots, a_{m+k-1})$ depend on $a, a_{m+1}, \ldots, a_{m+k-1}$ and

$$J_k^{\pm}(a) = \int_{L_\pm} \frac{u^2 \mathrm{d}u}{(1 - u/\lambda_1)^{\nu_1} (1 - u/\lambda_2)^{\nu_2}},$$

(for $a^2 \neq 8$).

Lemma The only solutions to the system

$$c^+(a) = c^+(\tilde{a}), \quad c^-(a) = c^-(\tilde{a})$$

are of the form (a, a), $a \in \mathbb{C}$.

Lemma If Re$a \leq 0$ then $J_k^-(a) \neq 0$ and if Re$a \geq 0$ then $J_k^+(a) \neq 0$.

We choose the following ways of prolongation of the local first integral:

along L_- *for* Re$a \leq 0$ *and along* L_+ *for* Re$a > 0$.

Proposition Depending on the way of prolongation, the local first integral $\boldsymbol{H}$ has the expansion

$$\boldsymbol{H}(0, z) = c^{\pm} z^2 + \sum_{k=1}^{\infty} c_k^{\pm} z^{1+k/m}$$

with $c_k^{\pm} = \delta_k^{\pm}(a)a_{m+k} + \tilde{c}_k(a_{m+1}, \ldots, a_{m+k-1})$, $\delta_k^{\pm}(a) \neq 0$, polynomials in $a_{m+1}, \ldots, a_{m+k}$ with coefficients depending on $a = a_m$, which are coefficients of the Puiseux expansion of the curve $x \longmapsto (X, Y) = \left(A(x), x^{2m}\right)$ at $(X, Y) = (0, 0)$, i.e.,

$$X = \sum_{k \geq m} a_k Y^{k/(2m)}.$$

This proposition gives a generalization of the Taylor expansions of the integral at infinity at the boundary lines $\mathbb{D}_0$ and $\mathbb{D}_1$. But now the coefficients $\delta_k^{\pm}$ before a_{m+k} in $c_k^{\pm}$ depend on $a = a_m$. These coefficients may be different for $x = 0$ and for $x = 1$, a and $\tilde{a}$ respectively. But when we consider two variants of connection of points $(0, z)$ with $(x, 0)$ (with correspondingly different local cut sets), respectively, $(z, 1)$ with $(\tilde{x}, 0)$ then the first lemma implies $a = \tilde{a}$. This leads to the assumption of the Lüroth theorem.

Thank you very much for your attention!