Exponential growth order of solutions for Moser irreducible system and a counterexample for Barkatou's conjecture

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We study the maximal exponential growth order of solutions of the following singular system of ordinary differential equations of apparent Poincaré rank $p \geq 1$ under the Moser irreducibility condition,

$$Ly = 0, \quad L = (p, A(z)) := z^{p+1}(d/dz)I_N - A(z), \tag{1}$$

where $A(z) \in M_N(\mathbb{C}\{z\})$ with $A(z) = \sum_{n=0}^{\infty} A_n z^{k+n}$, $k = O(A) \geq 0$ (order of zeros), $r = \operatorname{rank} A_0 \geq 1$.

In 1960, J. Moser defined the notion of irreducibility as follows. Let define two numbers $m(A)$ and $\mu(A)$ by

$$m(A) = p - k + r/N,$$
$$\mu(A) = \min_{P(z) \in GL_N(\mathbb{K}[z])} \left\{ m(A_P) \; ; \; A_P := P^{-1}AP - z^{p+1}P^{-1}P'(z) \in M_N(\mathbb{C}\{z\}) \right\}.$$

Then, in the case when $m(A) > 1$, he called the operator L reducible if $m(A) > \mu(A)$, and irreducible otherwise. He characterized the irreducibility condition in the following way; *The operator $L = (p, A(z))$ is irreducible if and only if*

$$\mathcal{P}_A(\lambda) := \left[z^r \times \left(\det(\lambda I_N - A(z)/z^{k+1}) \right) \right]_{z=0} \not\equiv 0. \tag{2}$$

We denote by $\rho(L)$ the maximal exponential growth order of solutions of the equation (1). Then we can prove the following estimate for $\rho(L)$; *Let the operator $L = (p, A(z))$ be Moser irreducible with a nilpotent constant term $A(0) = A_0$. Then $\rho(L)$ is estimated by*

$$p - \frac{N - d - r}{N - d} \leq \rho(L) \leq p - \frac{1}{k_1}, \quad d = \deg_\lambda \mathcal{P}_A(\lambda), \; k_1 = \min\{k \; ; \; A_0^k = O\}. \tag{3}$$

The best possibility of this estimate will be shown by an example. We remark that the first inequality is found in his lecture by M. Barkatou without proof [p.31, see below].

Of course, it is desirable to specify $\rho(L)$ exactly, if possible. M. Barkatou stated a result on this problem in the lecture under some condition [p.33], and gave a conjecture that; *For Moser irreducible operator $L = (p, A(z))$ it may hold that*

$$\rho(L) = p - s_0(A),$$
$$s_0(A) := \min_{1 \leq j \leq N} \frac{O(p_j)}{j}, \quad p_A(\lambda, z) = \det(\lambda I_N - A(z)) = \sum_{j=0}^{N} p_j(z)\lambda^{N-j}, \tag{4}$$

in the tutorial lecture at the conference ISSAC'10 in Münich [p.35]. We note that in the case when $s_0(A) = 0$ the assertion is trivial since $s_0(A) = 0$ if and only if the constant term A_0 is not nilpotent.

In the lecture, we give a counterexample to this conjecture, and present some properties of system transformation which makes change the Jordan canonical form of A_0 in Moser irreducible case.