Diophantine approximation of fractional parts of powers of real numbers

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Outline

1. Distribution of fractional parts of powers of real numbers
2. Some known results in Diophantine approximation
3. Diophantine approximation of $\{x^n\}$
4. Proofs on the uniform Diophantine approximation of $\{x^n\}$

I. Equidistribution

A sequence (u_n) in $[0,1]$ is equidistributed if for all interval $[a,b] \subset [0,1]$,

$$\lim_{N\to\infty} \frac{\text{Card}\{1 \le n \le N : u_n \in [a,b]\}}{N} = b - a.$$

Theorem (Weyl, 1916) :
A sequence (u_n) is equidistributed if and only if for every complex-valued, 1-periodic continuous function f,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(u_n) = \int_0^1 f(x)dx,$$

and, if and only if for all integer $h \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2i\pi h u_n} = 0.$$

II. (Equi)-Distribution of $\{x^n\}$

Denote $\{\cdot\}$ the fractional part of a real number.

Weyl 1916 : Let $x > 1$ be a real number. Then for almost all real ξ, the sequence $\{\xi x^n\}$ is equidistributed.

Koksma 1935 : Let $\xi \neq 0$ be a real number. Then for almost all real $x > 1$, the sequence $\{\xi x^n\}$ is equidistributed.

Denote by $\|\cdot\|$ the distance to the nearest integer.

Thue 1910 (Hardy 1919) : Let $\xi \neq 0$ and $x > 1$ be two real numbers. If there exist real numbers $C > 0$ and $0 < \rho < 1$ such that $\|\xi x^n\| < C\rho^n$ for all $n \geq 1$, then x is an algebraic number.

Pisot 1937 : Let $\xi \neq 0$ and $x > 1$ be two real numbers such that

$$\sum_{n=0}^{\infty} \|\xi x^n\|^2 < \infty.$$

Then $\xi \in \mathbb{Q}(x)$ and x is a Pisot-Vijayaraghavan number : an algebraic integer > 1, whose Galois conjugates have module < 1.

III. Sizes of exceptional sets

Pollington 1979 : Let $x > 1$ be a real number. The set of numbers ξ such that $\{\xi x^n\}$ is not dense (so not equidistributed), has Hausdorff dimension 1.

Pollington 1980 : Let $\xi \neq 0$ be a real number. For all $\delta > 0$, the set

$$\left\{x > 1 : \{\xi x^n\} \in [0, \delta] \text{ for all } n \geq 1\right\}$$

has Hausdorff dimension 1. Thus, the set of numbers $x > 1$ such that $\{\xi x^n\}$ is not dense (so not equidistributed), has Hausdorff dimension 1.

Remark : **Vijayaraghavan 1948** proved that for all $\delta > 0$, there are uncountably many $x > 1$, such that $\|x^n\| \leq \delta$ for all $n \geq 1$.

IV. Sizes of exceptional sets - continued

Bugeaud–Moshchevitin 2012, Kahane 2014 : Let (b_n) be an arbitrary sequence in $[0, 1]$, and $\delta > 0$. The set

$$\left\{x > 1 : \|x^n - b_n\| \leq \delta \ \text{ for all large } n\right\}$$

has Hausdorff dimension 1.

Kahane's question : for $X > \frac{1}{2\delta}$,

$$\dim_H \left\{1 < x < X : \|x^n - b_n\| \leq \delta \ \text{ for all large } n\right\} = ?$$

Candidate : $\log(2\delta X)/\log X$.

Bugeaud–L–Rams, in preparation : lower bound is OK.

V. A Number Theory motivation

Mahler 1957 : For sufficiently large k

$$\|(3/2)^k\| > (3/4)^{k-1}.$$

Then (**Waring's problem**) the number

$$g(k) := \min\{s \in \mathbb{N} : \text{ all } a \in \mathbb{N} \text{ can be written as } n_1^k + \cdots + n_s^k \text{ with } n_j \in \mathbb{N}\}$$

is

$$g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2.$$

Open problem : Is the sequence $\{(3/2)^k\}$ dense in $[0, 1]$?

Some known results

in Diophantine approximation

I. Dirichlet and Legendre

Denote by $\|\cdot\|$ the distance to the nearest integer.

Dirichlet Theorem, 1842 (uniform approximation) :
Let θ, Q be real numbers with $Q \geq 1$. There exists an integer n with $1 \leq n \leq Q$, such that

$$\|n\theta\| < Q^{-1}.$$

In other words,

$$\{\theta : \ \forall Q \geq 1, \ \|n\theta\| < Q^{-1} \text{ has a solution } 1 \leq n \leq Q\} = \mathbb{R}.$$

Corollary (asymptotic approximation) :
For any real θ, there exist infinitely many integers n such that

$$\|n\theta\| < n^{-1}.$$

In other words,

$$\{\theta : \ \|n\theta\| < n^{-1} \text{ for infinitely many } n\} = \mathbb{R}.$$

Legendre 1808 "Essai sur la théorie des nombres" : proved the asymptotic approximation property by using continued fractions.

II. Approximation with a higher speed

Jarník 1929, Besicovith 1934 : For $w > 1$, the Hausdorff dimension

$$\dim_H(\mathcal{L}_w) = \dim_H \left\{\theta : \ \|n\theta\| < n^{-w} \text{ i.o. } n\right\} = 2/(w+1).$$

What is about the set

$$\mathcal{U}_w := \left\{\theta : \ \forall Q > 1, \ \|n\theta\| < Q^{-w} \text{ has a solution } 1 \le n \le Q\right\}?$$

Khintchine 1926 : For $w > 1$, $\mathcal{U}_w$ is empty.

Proof : Apply the continued fraction theory.

III. Question on inhomogeneous terms -1

Bugeaud 2003, Troubetzkoy–Schmeling 2003 : for all $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $w \geq 1$, set

$$\mathcal{L}_w[\theta] := \{y : ||n\theta - y|| < n^{-w} \text{ for infinitely many } n\}.$$

Then

$$\dim_H(\mathcal{L}_w[\theta]) = 1/w.$$

Liao–Rams 2013 : Sharp estimations for general speed : $n^{-w} \to \phi(n)$.

Question of **Bugeaud–Laurent 2005** : for a fixed irrational θ, what is the size (Hausdorff dimension) of the set

$$\mathcal{U}_w[\theta] := \left\{y : \ \forall Q \gg 1, \ \ \|n\theta - y\| < Q^{-w} \text{ has a solution } 1 \leq n \leq Q\right\}.$$

Remark :

$$\mathcal{U}_w[\theta] \setminus \{n\theta : n \in \mathbb{N}\} \subset \mathcal{L}_w[\theta].$$

IV. Question on inhomogeneous terms -2

For $\theta \notin \mathbb{Q}$, define $w(\theta) := \sup\{\beta > 0 : \liminf_{j\to\infty} j^\beta \|j\theta\| = 0\} \geq 1$.
Let $\{q_n\}$ be the denominators of continued fractions convergents of θ.

Theorem (D.H. Kim–L, arXiv 2015)

Let θ be an irrational with $w(\theta)$. Then the Hausdorff dimension of $\mathcal{U}_w[\theta]$ is 0 if $w > w(\theta)$, is 1 if $w < 1/w(\theta)$, and equals to

$$\begin{cases} \underline{\lim}_{k\to\infty} \dfrac{\log(\prod_{j=1}^{k-1}(n_j^{1/w}\|n_j\theta\|)\cdot n_k^{1/w+1})}{\log(n_k\|n_k\theta\|^{-1})}, & \dfrac{1}{w(\theta)} < w < 1, \\ \underline{\lim}_{k\to\infty} \dfrac{-\log(\prod_{j=1}^{k-1} n_j\|n_j\theta\|^{1/w})}{\log(n_k\|n_k\theta\|^{-1})}, & 1 < w < w(\theta). \end{cases}$$

where n_k is the (maximal) subsequence of (q_k) such that

$$\begin{cases} n_k^{1/w}\|n_k\theta\| < 1, & \text{if } 1/w(\theta) < w < 1, \\ n_k\|n_k\theta\|^{1/w} < 2, & \text{if } 1 < w < w(\theta). \end{cases}$$

V. Diophantine approximation of β-transformation -1

Let $\beta > 1$ be a real number and T_β be the β-transformation defined by

$$\text{for } x \in [0,1], \quad T_\beta x = \beta x \bmod 1.$$

Let $v_\beta(x)$ be the supremum of the real numbers v such that

$$T_\beta^n(x) < (\beta^n)^{-v}, \ i.o. \ n.$$

Shen–Wang 2013 :

$$\dim\{x \in [0,1] : v_\beta(x) \geq v\} = \frac{1}{1+v}.$$

Persson–Schmeling 2008 :

$$\dim\{\beta > 1 : v_\beta(1) \geq v\} = \frac{1}{1+v}.$$

VI. Diophantine approximation of β-transformation -2

Let $\hat{v}_\beta(x)$ be the supremum of the real numbers $\hat{v}$ such that

$$\forall N \gg 1, T_\beta^n(x) < (\beta^N)^{-\hat{v}} \text{ has a solution } 1 \le n \le N.$$

Theorem (Bugeaud–L, to appear)

Let θ and $\hat{v}$ be positive real numbers with $\hat{v} < 1$ and $\theta \ge 1/(1-\hat{v})$, then

$$\dim(\{x : \hat{v}_\beta(x) = \hat{v}\} \cap \{x : v_\beta(x) = \theta\hat{v}\}) = \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)},$$

$$\dim(\{\beta > 1 : \hat{v}_\beta(1) = \hat{v}\} \cap \{\beta > 1 : v_\beta(1) = \theta\hat{v}\}) = \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)},$$

$$\dim\{x : \hat{v}_\beta(x) \ge \hat{v}\} = \dim\{x : \hat{v}_\beta(x) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2,$$

$$\dim\{\beta > 1 : \hat{v}_\beta(1) \ge \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2.$$

VII. Relation with the hitting time

Let $(T_\theta)_{\theta\in\Theta}$ $(\Theta\subset\mathbb{R})$ be a family of systems on a metric space (X,d). Define

$$\tau_r^\theta(x,y)=\inf\{n : T_\theta^n x\in B(y,r)\}.$$

and define (for the zero entropy systems)

$$\underline{R}^\theta(x,y):=\liminf_{r\to 0}\frac{\log\tau_r^\theta(x,y)}{-\log r},\qquad \overline{R}^\theta(x,y):=\limsup_{r\to 0}\frac{\log\tau_r^\theta(x,y)}{-\log r}.$$

We have (fixing $x,y\in X$)

$$\mathcal{L}_w=\{\theta : d(T_\theta^n x,y)<n^{-w}\ \ i.o.\}\approx\{\theta : \underline{R}^\theta(x,y)\le 1/w\},$$

$$\mathcal{U}_w=\{\theta : \forall N\gg 1, \exists\ 1\le n\le N, d(T_\theta^n x,y)<N^{-w}\}\approx\{\theta : \overline{R}^\theta(x,y)\le 1/w\}.$$

Thus, $\mathcal{U}_w$ is almost less than $\mathcal{L}_w$.

$\rightarrow$ The same thing holds when fixing (θ,x) or (θ,y).

$\rightarrow$ Positive entropy systems, sometimes : replace $\log\tau_r(x,y)$ by $\tau_r(x,y)$.

VIII. Level sets of hitting time

Shrinking target problem : Fix one dynamical system T, fix one point y, one studies the size of the level set

$$\{x \in X : \underline{R}(x, y) = \alpha\}, \quad \text{for a given } \alpha.$$

Measure results : **Boshernitzan, Chernov, Chazottes, Fayad, Galatalo, Kleinbock, Kim...**
Hausdorff dimension results : **Hill–Velani 1995, 1999 ; Urbański 2002 ; Fernández–Melián–Pestana 2007 ; Shen–Wang 2013 ; Li–Wang–Wu–Xu 2014, Bugeaud–Wang 2014**.

For sets of parameters : **Persson-Schmeling 2008, Li-Persson-Wang-Wu 2014, Aspenberg-Persson**

Dynamical diophantine approximation problem : Fix one dynamical system T, fix one point x, one studies the size of the level set

$$\{y \in X : \underline{R}(x, y) = \alpha\}, \quad \text{for a given } \alpha.$$

Fan–Schmeling-Troubetzkoy 2013 ; Liao–Seuret 2013 ; Persson–Rams 2015.

Diophantine approximation of $\{x^n\}$

I. Measure result - asymptotic approximation of $\{x^n\}$

Koksma 1945 : Let (ϵ_n) be a real sequence with $0 \leq \epsilon_n \leq 1/2$ for all $n \geq 1$. If $\sum \epsilon_n < \infty$, then for almost all $x > 1$,

$$\|x^n\| \leq \epsilon_n \quad \text{only for finitely many } n.$$

If (ϵ_n) is **non-increasing** and $\sum \epsilon_n = \infty$, then for almost all$x > 1$

$$\|x^n\| \leq \epsilon_n \quad \text{for infinitely many } n.$$

Mahler–Szekeres 1967 : for almost $x > 1$,

$$\lim_{n \to \infty} \|x^n\|^{1/n} = 1.$$

II. Dimension result -asymptotic approximation of $\{x^n\}$

For $x > 1$, put $P(x) := \liminf_{n\to\infty} \|x^n\|^{1/n}$.

Mahler–Szekeres 1967 : "$P(x) = 0$" $\Rightarrow$ x is transcendental.

Remark that for $b > 1$,

$$\{x > 1 : P(x) < 1/b\} = \{x > 1 : \|x^n\| < b^{-n} \text{ for infinitely many } n\}.$$

Question : What is the size of $\{x > 1 : P(x) < 1/b\}$?
Bugeaud–Dubickas 2008 : For all real number $X > 1$, and $b > 1$,

$$\dim_H \{1 < x < X : P(x) < 1/b\} = \frac{\log X}{\log(bX)}.$$

Moreover, $\dim_H \{x > 1 : P(x) < 1/b\} = 1$.

Proof : Mass transference principle (**Beresnevich–Velani 2006**) :

$$Leb(\limsup B(x_n, r_n)) = 1 \Rightarrow \mathcal{H}^s(\limsup B(x_n, r_n^{1/s})) = \infty.$$

III. Dimension result - asymptotic case (continued)

Bugeaud–L–Rams, in preparation : a constructive proof for the result of Bugeaud–Dubickas. In general, for an arbitrary real sequence (y_n),

$$\dim_H \left\{1 < x < X : \|x^n - y_n\| < b^{-n} \text{ for infinitely many } n\right\} = \frac{\log X}{\log(bX)}.$$

Thus for a sequence (z_k),

$$\dim_H \bigcap_{k=1}^{\infty} \left\{1 < x < X : \|x^n - z_k\| < b^{-n} \text{ for infinitely many } n\right\} = \frac{\log X}{\log(bX)}.$$

Hence, we also have

$$\dim_H \bigcap_{k=1}^{\infty} \left\{x > 1 : \|x^n - z_k\| < b^{-n} \text{ for infinitely many } n\right\} = 1.$$

IV. Uniform approximation question

For $y \in \mathbb{R}$, we are interested in the following uniform approximation

$$E(y,\tau) = \Big\{x > 1 : \forall N \gg 1, \|x^n - y\| \le \tau^{-N} \text{ has a solution } 1 \le n \le N\Big\}.$$

Furthermore, we also study

$$E(y,\tau,b) := \{x > 1 : \|z^n - y\| < b^{-n} \text{ for infinitely many } n$$
$$\text{and } \forall N \gg 1, ||z^n - y|| < \tau^{-N} \text{ has a solution } 1 \le n \le N\}.$$

Question : Hausdorff dimensions of $E(y,\tau)$ and $E(y,\tau,b)$?

V. Results on uniform approximation of $\{x^n\}$

Theorem (Bugeaud–L–Rams, in preparation)

Suppose $b = \tau^\theta$ with $\theta > 1$. Then for all $y \in \mathbb{R}$,

$$\dim_H \left(E(y, \tau, b) \cap]1, X[\right) \geq \frac{\log X - \frac{\theta}{\theta - 1} \log \tau}{\log X + \theta \log \tau}.$$

Maximizing with respect to θ ($\theta = 2 \log X / (\log X - \log \tau)$), we have

$$\dim_H \left(E(y, \tau) \cap]1, X[\right) \geq \left(\frac{\log X - \log \tau}{\log X + \log \tau} \right)^2.$$

Corollary (Bugeaud–L–Rams, in preparation)

For all $y \in \mathbb{R}$, and all $b \geq \tau > 1$,

$$\dim_H E(y, \tau, b) = \dim_H E(y, \tau) = 1.$$

Proofs on the uniform Diophantine approximation of $\{x^n\}$

I. Construction of a subset

We will construct a subset F of the set

$$E(X, y, \tau, b) := \{1 < x < X : \|z^n - y\| < b^{-n} \text{ for infinitely many } n \\ \text{and } \forall N \gg 1, ||z^n - y|| < \tau^{-N} \text{ has a solution } 1 \leq n \leq N\}.$$

Suppose $b = \tau^\theta$ with $\theta > 1$. **Take** $n_k = \lfloor \theta^k \rfloor$. Consider the points $1 < z < X$ such that

$$||z^{n_k} - y|| < b^{-n_k} \Leftrightarrow \exists m, \quad z \in \left[\left(m + y - \frac{1}{b^{n_k}}\right)^{\frac{1}{n_k}}, \left(m + y + \frac{1}{b^{n_k}}\right)^{\frac{1}{n_k}} \right].$$

Then $z \in E(X, y, \beta, b) = E(X, y, \beta, \tau^\theta)$.

- Level 1 : $I_{n_1}(m, y, b) := [(m + y - b^{-n_1})^{1/n_1}, (m + y + b^{-n_1})^{1/n_1}]$, where $m \in]1, X^{n_1}[$ is an integer.
- Level 2 : for an interval $[c, d]$ at level 1, its son-intervals are $I_{n_2}(m, y, b) := [(m + y - b^{-n_2})^{1/n_2}, (m + y + b^{-n_2})^{1/n_2}]$ with $m \in [c^{n_2}, d^{n_2}]$.

II. Computer the numbers and the lengths

By construction, for the fundamental intervals containing $z \in F$, we have

- each interval $[c_k, d_k]$ at level k contains at least

$$m_k(z) \approx d_k^{n_{k+1}} - c_k^{n_{k+1}} \geq n_{k+1} c_k^{n_{k+1}-1} \cdot \frac{2}{n_k b^{n_k} d_k^{n_k - 1}} \approx \left(\frac{z^{\theta-1}}{b} \right)^{\theta^k}$$

  son-intervals at level $k+1$.
- the distance between intervals at level $k+1$ contained in the interval $[c_k, d_k]$ at level k is at least

$$\epsilon_k(z) = \frac{1 - 2/b^{n_{k+1}}}{n_{k+1} d_k^{n_{k+1}-1}} \approx \frac{1}{z^{\theta^{k+1}}}.$$

III. Local dimension

The local dimension of $z \in F$ is bounded from below by

$$\frac{\log(m_1(z)\cdots m_{k-1}(z))}{-\log m_k(z)\epsilon_k(z)} \approx \frac{\big((\theta-1)\log z - \log b\big)\sum_{j=1}^{k-1}\theta^k}{-\log\theta^k \log bz}$$
$$= \frac{(\theta-1)\log z - \log b}{(\theta-1)\log bz}\cdot\frac{\theta^k-1}{\theta^k}$$
$$\geq \frac{(\theta-1)\log z - \log b}{(\theta-1)\log bz} - \varepsilon'(k)$$

with $\varepsilon'(k) \to 0$ when $k \to \infty$. Using the relation $b = \tau^\theta$, we have a lower bound of the dimension of $E(X, y, \tau, b)$:

$$\frac{(\theta-1)\log X - \log b}{(\theta-1)\log bX} = \frac{\log X - \frac{\theta}{\theta-1}\log\tau}{\log X + \theta\log\tau}.$$