Self-similar sets from the topological point of view

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joint work with T. Banakh, F. Strobin

Self-similar sets

X - topological space
$\mathcal{H}(X)$ - the space of nonempty, compact subsets of X

Definition

For a dynamical system on $\mathcal{H}(X)$ generated by a finite family $\mathcal{F}$ of continuous maps $X \to X$, such that

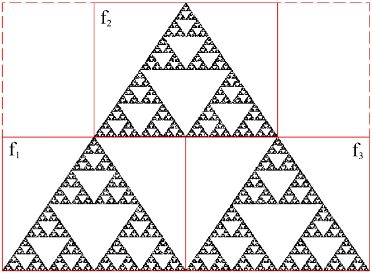
$$K \in \mathcal{H}(X) \qquad \mathcal{F}(K) = \bigcup_{f \in \mathcal{F}} f(K),$$

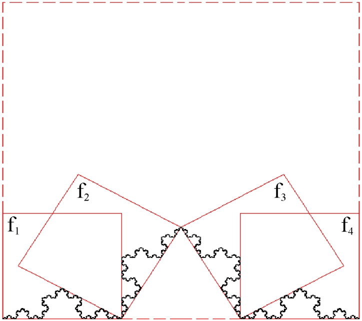
the **self-similar set** (**fractal**) is a nonempty compact set $A \subset X$ such that $A = \mathcal{F}(A)$ and for every compact set $K \in \mathcal{H}(X)$ the sequence $(\mathcal{F}^n(K))_{n=1}^{\infty}$ converges to A in the Vietoris topology on $\mathcal{H}(X)$.

Classical self-similar sets

Definition

For a complete metric space X and a family $\mathcal{F}$ of Banach contractions, the self-similar set is called the attractor of iterated function system (IFS) $\mathcal{F}$ or **IFS-attractor**.





Definition

A compact space $A = \bigcup_{f \in \mathcal{F}} f(A)$ for continuous $f\colon A \to A$ is

- **topological fractal** if A is a Hausdorff space and each $f \in \mathcal{F}$ is *topologically contracting*; for every open cover $\mathcal{U}$ of A there is $n \in \mathbb{N}$ such that for any maps $f_1, \dots, f_n \in \mathcal{F}$ the set $f_1 \circ \cdots \circ f_n(A) \subset U \in \mathcal{U}$.
- **Banach fractal** if A is homeomorphic to some IFS-attractor.
- **Euclidean fractal** if A is homeomorphic to some IFS-attractor in $\mathbb{R}^n$.
- **Banach ultrafractal** if A is metrizable, the family $(f(A))_{f \in \mathcal{F}}$ is disjoint and for any $\lambda > 0$ each $f \in \mathcal{F}$ has $\mathrm{Lip}(f) < \lambda$ with respect to some *ultrametric* generating the topology of A.

A metric d on X is called an *ultrametric* if it satisfies the strong triangle inequality $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for $x, y, z \in X$.

Fact 1

For any compact metrizable space we have the implications

$$\text{Banach ultrafr.} \Rightarrow \text{Euclidean fr.} \Rightarrow \text{Banach fr.} \Rightarrow \text{topological fr.}$$

Fact 2

The topology of a compact metrizable space X is generated by an ultrametric if and only if X is zero-dimensional (has a base of closed-and-open sets).

$$\text{Banach ultrafractal} \Rightarrow \text{zero-dimensional space}$$

Problem

Which zero-dimensional compact spaces are Banach ultrafractals?

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Scattered height

For a topological space X let

$$X' = \{x \in X \colon x \text{ is an accumulation point of } X\}$$

be the *Cantor-Bendixson derivative* of X.

- $X^{(\alpha+1)} = (X^{(\alpha)})'$
- $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ for a limit ordinal α

Definition

For a countable compact topological space X we define its height

$$\hbar(X) = \min\{\beta \colon X^{(\beta)} \text{ is finite}\}.$$

For an uncountable space X we put $\hbar(X) = \infty$, where $\infty > \alpha$ for each ordinal α.

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Theorem (Banakh,N.,Strobin) 2014

For a zero-dimensional compact metrizable space X the following conditions are equivalent:

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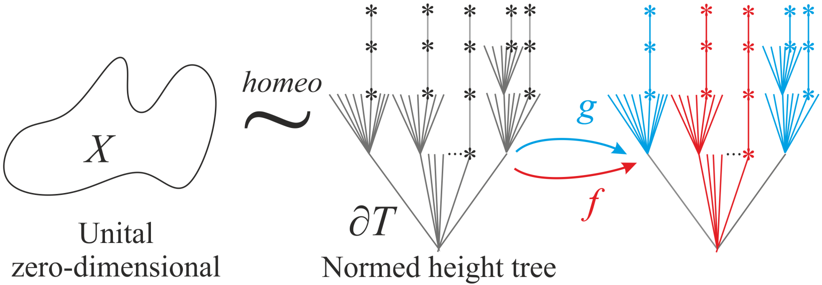
Each compact metrizable space can be written as a finite topological sum of its unital subspaces.

The compact zero-dimensional unital space X with non-limit $\hbar(X)$ is a closure of disjoint union $X = \overline{\bigcup_{i\in\mathbb{N}} X_i}$ of unital spaces of the same height $\hbar(X_i) + 1 = \hbar(X)$, where $\infty + 1 = \infty$.

Idea of the proof



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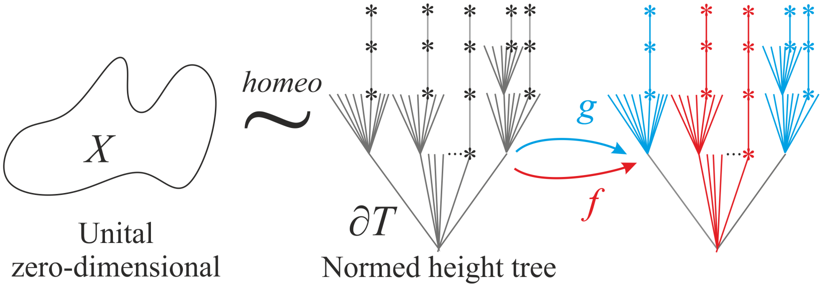

Lemma

For any zero-dimensional unital spaces T, S with $\hbar(T) \geqslant \hbar(S)$ there exists a continuous surjection $f : T \to S$.

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A metric d on the space X is called **doubling** if there exists a natural number M such that each open ball $B(x, r)$ is contained in at most M open balls $B(y, \frac{r}{2})$.

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Assouad's theorem

For each metric space X with doubling metric d and for each $\alpha \in (0, 1)$ there exists bi-Lipschitz function $f : (X, d^\alpha) \to \mathbb{R}^n$ which embeds space (X, d^α) into the Euclidean space.

Euclidean fractals

Lemma

Each IFS-attractor with doubling metric is Euclidean fractal.

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Let $k \in \mathbb{N}$ be such that
$\mathrm{Lip}(\varphi) \cdot (\lambda^\alpha)^k \cdot \mathrm{Lip}(\varphi^{-1}) < 1$

$$\mathcal{F}_{\mathbb{R}^n} = \{\varphi \circ f_1 \circ \cdots \circ f_k \circ \varphi^{-1} : f_i \in \mathcal{F}\}$$

IFS-attractors with doubling metric

Fact

Bi-Lipschitz image of IFS-attractor is also IFS-attractor.

IFS-attractors with doubling metric

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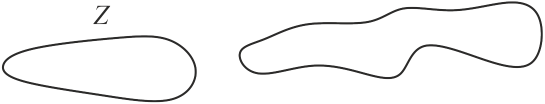
Problem

Which compact spaces are homeomorphic to IFS-attractor with doubling metric?

Compact spaces as Euclidean fractals

Theorem (Banakh, N 2015)

Let $X \subset \mathbb{R}^n$ be compact set and Z be its uncountable, zero-dimensional, open subset. Then X is an Euclidean fractal.



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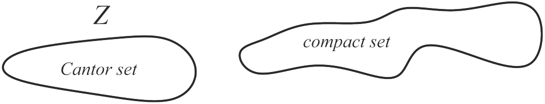

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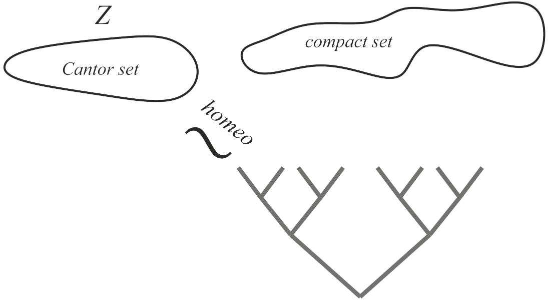

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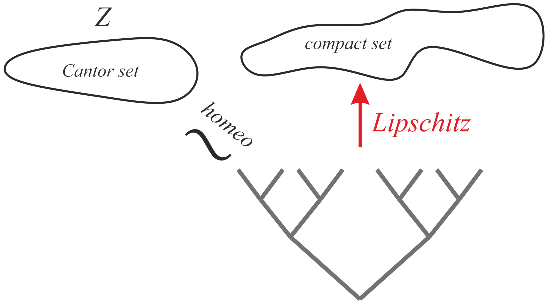

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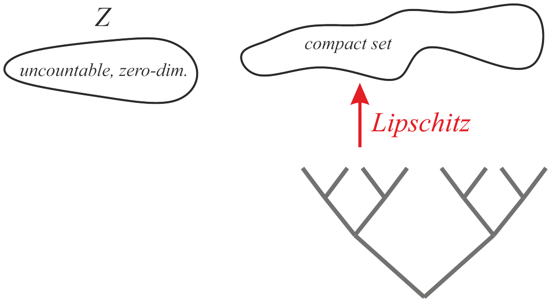

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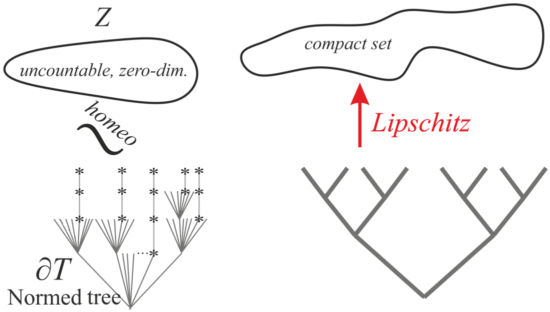

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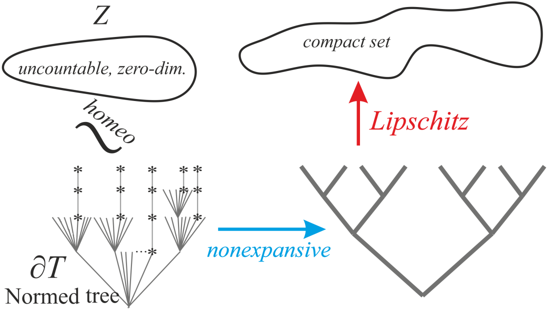

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