Heat kernel estimates for subordinate Brownian motions

Ante Mimica

Department of Mathematics
University of Zagreb
Croatia

Probability and Analysis
Będlewo
May 5, 2015

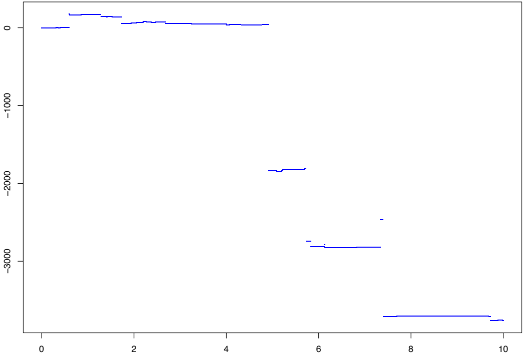
$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t\rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

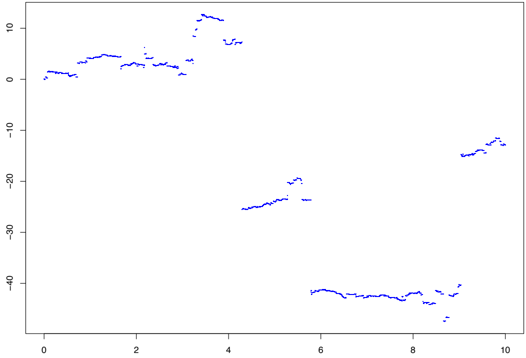
$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t\rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$



$\alpha = 0.5$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

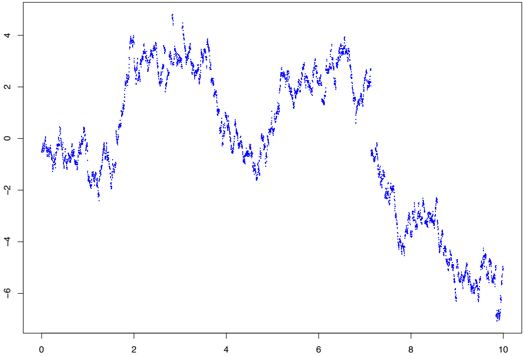
$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t\rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$



$\alpha = 1.0$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

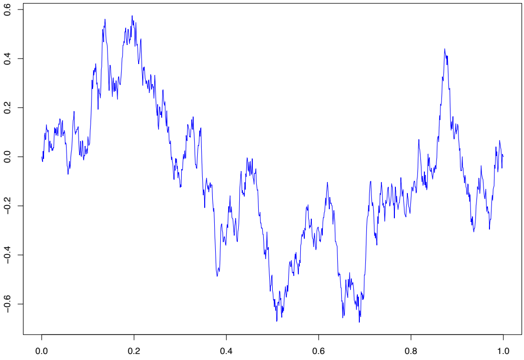
$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t\rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$



$\alpha = 1.8$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t \rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$



$\alpha = 2$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t\rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$

- scaling

$$p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x)$$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t\rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$

- scaling

$$p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x)$$

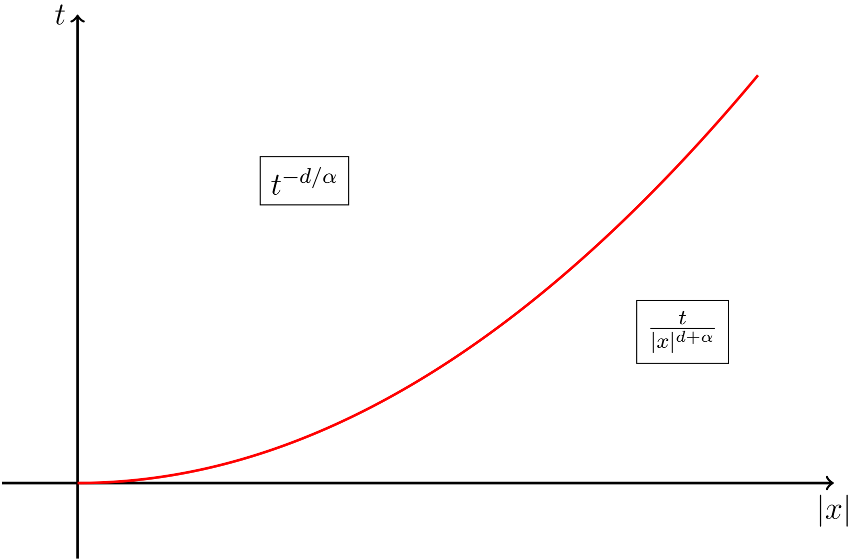
- Polya '23, Blumenthal-Getoor '60

$$|x|^{d+\alpha} p(1,x) \underset{|x|\to\infty}{\longrightarrow} c_{d,\alpha} \sin \tfrac{\alpha\pi}{2}$$

$\{X_t : t \geq 0\}$ rotationally invariant α-stable process in $\mathbb{R}^d$

$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{i\langle \xi, X_t \rangle}] = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p(t,x)\, dx, \quad 0 < \alpha \leq 2$$

- scaling
$$p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x)$$
- Polya '23, Blumenthal-Getoor '60
$$|x|^{d+\alpha} p(1,x) \underset{|x|\to\infty}{\longrightarrow} c_{d,\alpha} \sin \tfrac{\alpha\pi}{2}$$
- $p(t,x) \asymp t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad 0 < \alpha < 2$



$$t|x|^{-\alpha} = 1$$

Subordinator

A Lévy process $S = \{S_t : t \geq 0\}$ taking values in $[0, \infty)$.

Subordinator

A Lévy process $S = \{S_t : t \geq 0\}$ taking values in $[0, \infty)$.

- Laplace transform $\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \ \lambda \geq 0$

Subordinator

A Lévy process $S = \{S_t : t \geq 0\}$ taking values in $[0, \infty)$.

- Laplace transform $\qquad \mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \ \lambda \geq 0$
- Laplace exponent

$$\phi(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt)$$

$$\underbrace{d \geq 0}_{\text{drift}}, \qquad \int_{(0,\infty)} (1 \wedge t) \underbrace{\mu(dt)}_{\substack{\text{Lévy} \\ \text{measure}}} < \infty$$

Subordinator

A Lévy process $S = \{S_t : t \geq 0\}$ taking values in $[0, \infty)$.

- Laplace transform $\quad \mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \ \lambda \geq 0$
- Laplace exponent

$$\phi(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt)$$

$$\underbrace{d \geq 0}_{\text{drift}}, \qquad \int_{(0,\infty)} (1 \wedge t) \underbrace{\mu(dt)}_{\substack{\text{Lévy} \\ \text{measure}}} < \infty$$

- ϕ is a Bernstein function: $\phi \geq 0$ and

$$(-1)^{n+1}\phi^{(n)}(\lambda) \geq 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > 0$$

$B = \{B_t : t \geq 0\}$ Brownian motion in $\mathbb{R}^d$ independent of S

$B = \{B_t : t \geq 0\}$ Brownian motion in $\mathbb{R}^d$ independent of S

Subordinate Brownian motion

Process $X = \{X_t : t \geq 0\}$ defined by

$$X_t = B_{S_t}, \;\; t \geq 0 \,.$$

$B = \{B_t : t \geq 0\}$ Brownian motion in $\mathbb{R}^d$ independent of S

Subordinate Brownian motion

Process $X = \{X_t : t \geq 0\}$ defined by

$$X_t = B_{S_t}, \;\; t \geq 0 \, .$$

- X Lévy process

$$\mathbb{E}[e^{i\langle \xi, X_t \rangle}] = e^{-t\phi(|\xi|^2)}$$

$B = \{B_t : t \geq 0\}$ Brownian motion in $\mathbb{R}^d$ independent of S

Subordinate Brownian motion

Process $X = \{X_t : t \geq 0\}$ defined by

$$X_t = B_{S_t}, \ \ t \geq 0\,.$$

- X Lévy process

$$\mathbb{E}\big[e^{i\langle \xi, X_t\rangle}\big] = e^{-t\phi(|\xi|^2)}$$

- transition density $\quad \mathbb{P}_x(X_t \in A) = \int_A p(t,x,y)\,dy$

$$p(t,x,y) = p(t,y-x) = \int_{(0,\infty)} (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4s}} \mathbb{P}(S_t \in ds)$$

Rotationally invariant α-stable process revisited

Rotationally invariant α-stable process revisited

S $\alpha/2$-stabe subordinator $0 < \alpha < 2$

Rotationally invariant α-stable process revisited

S $\alpha/2$-stabe subordinator $0 < \alpha < 2$

$$\phi(\lambda) = \lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t^{1+\alpha/2}}$$

Rotationally invariant α-stable process revisited

S $\alpha/2$-stabe subordinator $0 < \alpha < 2$

$$\phi(\lambda) = \lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t^{1+\alpha/2}}$$

- X is rotationally invariant α-stable process

Rotationally invariant α-stable process revisited

S $\alpha/2$-stabe subordinator $0 < \alpha < 2$

$$\phi(\lambda) = \lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t^{1+\alpha/2}}$$

- X is rotationally invariant α-stable process
- $p(t,x) \asymp t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} = \phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d}$

Rotationally invariant α-stable process revisited

S $\alpha/2$-stabe subordinator $0 < \alpha < 2$

$$\phi(\lambda) = \lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t^{1+\alpha/2}}$$

- X is rotationally invariant α-stable process
- $p(t,x) \asymp t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} = \phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d}$
- $\frac{\partial p}{\partial t}(t,x) = -(-\Delta)^{\alpha/2} p(t,x)$

Rotationally invariant α-stable process revisited

S $\alpha/2$-stabe subordinator $0 < \alpha < 2$

$$\phi(\lambda) = \lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{-\lambda t}) \frac{dt}{t^{1+\alpha/2}}$$

- X is rotationally invariant α-stable process
- $p(t,x) \asymp t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} = \phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d}$
- $\frac{\partial p}{\partial t}(t,x) = -(-\Delta)^{\alpha/2} p(t,x)$

$$-(-\Delta)^{\alpha/2} u(x) = \int_{\mathbb{R}^d} \left(u(x+y) - u(x) - \langle \nabla u(x), y \rangle 1_{\{|y| \le 1\}} \right) \underbrace{\frac{c_{d,\alpha}}{|y|^{d+\alpha}} dy}_{\text{Lévy measure}}$$

Recent results and examples...

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \leq 2$ '$\alpha = 0$'

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \le 2$ '$\alpha = 0$'
  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$

Recent results and examples...

- Bogdan, Grzywny, Ryznar ’14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \leq 2$ ’$\alpha = 0$’
  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$
  - $p(1, x) \underset{|x| \to 0+}{\longrightarrow} +\infty$ for $d > \beta$

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \leq 2$ '$\alpha = 0$'
  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$
  - $p(1,x) \underset{|x| \to 0+}{\longrightarrow} +\infty$ for $d > \beta$
- 'Dual' of geometric β-stable process '$\alpha = 2$'

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \leq 2$ — '$\alpha = 0$'
  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$
  - $p(1, x) \underset{|x| \to 0+}{\longrightarrow} +\infty$ for $d > \beta$
- 'Dual' of geometric β-stable process — '$\alpha = 2$'
  - $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})} - 1_{\{\beta=2\}}$

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \leq 2$ '$\alpha = 0$'
  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$
  - $p(1, x) \underset{|x| \to 0+}{\longrightarrow} +\infty$ for $d > \beta$
- 'Dual' of geometric β-stable process '$\alpha = 2$'
  - $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})} - 1_{\{\beta=2\}}$
  - Lévy measure of X is $J(y)dy$, where

$$J(y) \asymp \frac{1}{|y|^{d+2}(\log \frac{1}{|y|})^2}, \quad |y| \to 0+$$

Recent results and examples...

- Bogdan, Grzywny, Ryznar '14
  - estimates for unimodal convolution semigroups
  - weak scaling strictly between 0 and 2
- Geometric β-stable process, $0 < \beta \leq 2$ '$\alpha = 0$'
  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$
  - $p(1, x) \underset{|x| \to 0+}{\longrightarrow} +\infty$ for $d > \beta$
- 'Dual' of geometric β-stable process '$\alpha = 2$'
  - $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})} - 1_{\{\beta=2\}}$
  - Lévy measure of X is $J(y)dy$, where

$$J(y) \asymp \frac{1}{|y|^{d+2}(\log \frac{1}{|y|})^2}, \ |y| \to 0+$$

  - asymptotic expression for J not comparable to $\frac{\phi(|y|^{-2})}{|y|^d}$

Setting

For a function $f : (0, \infty) \to (0, \infty)$ the following conditions will be of interest

Setting

For a function $f : (0, \infty) \to (0, \infty)$ the following conditions will be of interest

Lower scaling condition (L)

There exist $\gamma > 0$, $\lambda_L \geq 0$ and $C_L > 0$ such that

$$\frac{f(\lambda x)}{f(\lambda)} \geq C_L x^\gamma \quad \text{for all} \quad \lambda > \lambda_L \;\text{ and }\; x \geq 1\,.$$

Setting

For a function $f : (0, \infty) \to (0, \infty)$ the following conditions will be of interest

Lower scaling condition (L)

There exist $\gamma > 0$, $\lambda_L \geq 0$ and $C_L > 0$ such that

$$\frac{f(\lambda x)}{f(\lambda)} \geq C_L x^{\gamma} \quad \text{for all} \quad \lambda > \lambda_L \quad \text{and} \quad x \geq 1\,.$$

Upper scaling condition (U)

There exist $\delta > 0$, $\lambda_U \geq 0$ and $C_U > 0$ such that

$$\frac{f(\lambda x)}{f(\lambda)} \leq C_U x^{\delta} \quad \text{for all} \quad \lambda > \lambda_U \quad \text{and} \quad x \geq 1\,.$$

- Model examples are functions f that vary regulary at infinity with index $\rho > 0$, i.e.

$$\lim_{\lambda\to\infty} \frac{f(\lambda x)}{f(\lambda)} = x^\rho \quad \text{for all} \quad \lambda > 0 \text{ and } \quad x > 0\,.$$

since for any $\varepsilon > 0$ there exists $C > 0$ and $\lambda_0 > 0$ such that

$$C^{-1}\left(\frac{t}{s}\right)^{\rho-\varepsilon} \le \frac{f(t)}{f(s)} \le C\left(\frac{t}{s}\right)^{\rho+\varepsilon},\ \lambda_0 \le s \le t$$

(Theorem of Potter).

- If f varies at 0 with index $\tilde{\rho} > 0$, then $\lambda_L = \lambda_U = 0$.
- $f(\lambda) = \log(1+\lambda)$ does not satisfy **(L)**
- Bernstein function ϕ always satisfies **(U)** with $\delta = 1$:

$$\phi(\lambda x) \le \phi(\lambda) x \quad \text{for all} \quad \lambda > 0 \text{ and } \quad x \ge 1\,.$$

We define a function $H : (0, \infty) \to (0, \infty)$ by

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t} - \lambda t e^{-\lambda t})\mu(dt)\,.$$

We define a function $H : (0, \infty) \to (0, \infty)$ by

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t} - \lambda t e^{-\lambda t})\mu(dt)\,.$$

Some properties of H (M. '15, see also Jain, Pruitt '87):

We define a function $H:(0,\infty)\to(0,\infty)$ by

$$H(\lambda):=\phi(\lambda)-\lambda\phi'(\lambda)=d\lambda+\int_{(0,\infty)}(1-e^{-\lambda t}-\lambda te^{-\lambda t})\mu(dt)\,.$$

Some properties of H (M. '15, see also Jain, Pruitt '87):

- if H satisfies **(U)** with $\delta<2$,

$$\mu(r,\infty)\asymp H(r^{-1}),\quad 0<r<r_0$$

We define a function $H:(0,\infty)\to(0,\infty)$ by

$$H(\lambda):=\phi(\lambda)-\lambda\phi'(\lambda)=d\lambda+\int_{(0,\infty)}(1-e^{-\lambda t}-\lambda t e^{-\lambda t})\mu(dt)\,.$$

Some properties of H (M. '15, see also Jain, Pruitt '87):

- if H satisfies **(U)** with $\delta<2$,

$$\mu(r,\infty)\asymp H(r^{-1}),\quad 0<r<r_0$$

- if H satisfies **(L)**,

$$H(\lambda)\asymp\lambda^2(-\phi''(\lambda)),\quad \lambda>\lambda_0$$

We define a function $H : (0, \infty) \to (0, \infty)$ by

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t} - \lambda t e^{-\lambda t})\mu(dt)\,.$$

Some properties of H (M. '15, see also Jain, Pruitt '87):

- if H satisfies **(U)** with $\delta < 2$,

$$\mu(r, \infty) \asymp H(r^{-1}), \quad 0 < r < r_0$$

- if H satisfies **(L)**,

$$H(\lambda) \asymp \lambda^2(-\phi''(\lambda)), \quad \lambda > \lambda_0$$

- ϕ satisfies **(U)** with $\delta < 1 \iff$

$$H(\lambda) \asymp \phi(\lambda), \quad \lambda > \lambda_U\,.$$

We define a function $H:(0,\infty)\to(0,\infty)$ by

$$H(\lambda):=\phi(\lambda)-\lambda\phi'(\lambda)=d\lambda+\int_{(0,\infty)}(1-e^{-\lambda t}-\lambda t e^{-\lambda t})\mu(dt)\,.$$

Some properties of H (M. '15, see also Jain, Pruitt '87):

- if H satisfies **(U)** with $\delta<2$,

$$\mu(r,\infty)\asymp H(r^{-1}),\quad 0<r<r_0$$

- if H satisfies **(L)**,

$$H(\lambda)\asymp\lambda^2(-\phi''(\lambda)),\quad \lambda>\lambda_0$$

- ϕ satisfies **(U)** with $\delta<1$ $\iff$

$$H(\lambda)\asymp\phi(\lambda),\quad \lambda>\lambda_U\,.$$

- $\frac{H(\lambda x)}{H(\lambda)}\le x^2$ for all $\lambda>0$ and $x\ge 1$

Main result

Theorem (Heat kernel estimates, M. '15)

Assume that ϕ *satisfies* **(L)** *and* H *satisfies* **(L)** *and* **(U)** *with* $\delta < 2$. *Then there exist constants* $\kappa_1, \kappa_2, a_L, a_U > 0$ *such that for* $0 < t < \kappa_1 \phi(\lambda_{L,\phi})^{-1}$ *and* $|x| < \kappa_2 \sqrt{\lambda_{L,H}^{-1} \wedge \lambda_{U,H}^{-1}}$

$$p(t,x) \asymp \begin{cases} \phi^{-1}(t^{-1})^{d/2} & t\phi(|x|^{-2}) > 1 \\ \frac{tH(|x|^{-2}) \vee e^{-a|x|^2 \phi^{-1}(t^{-1})}}{|x|^d} & t\phi(|x|^{-2}) \leq 1\,, \end{cases}$$

where $a = a_L$ *for the lower and* $a = a_U$ *for the upper bound.*

Main result

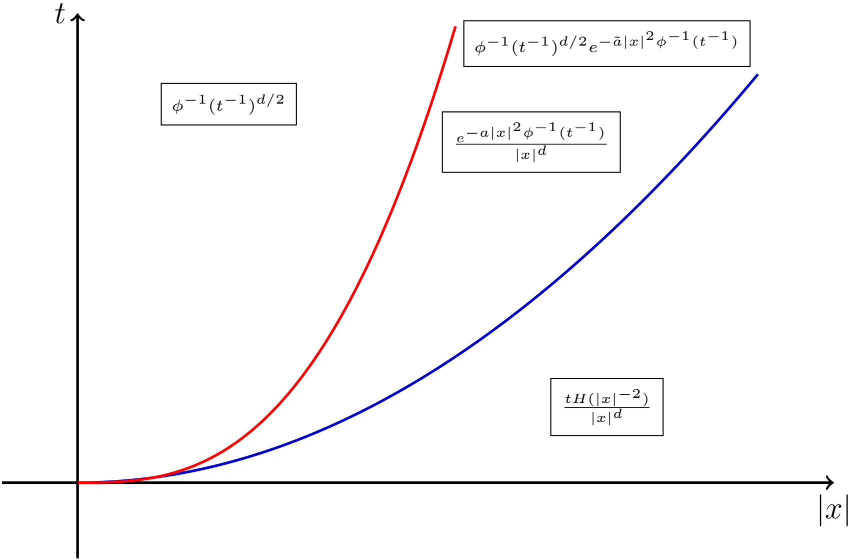
Theorem (Heat kernel estimates, M. '15)

Assume that ϕ satisfies **(L)** *and H satisfies* **(L)** *and* **(U)** *with $\delta < 2$. Then there exist constants $\kappa_1, \kappa_2, a_L, a_U > 0$ such that for $0 < t < \kappa_1 \phi(\lambda_{L,\phi})^{-1}$ and $|x| < \kappa_2 \sqrt{\lambda_{L,H}^{-1} \wedge \lambda_{U,H}^{-1}}$*

$$p(t,x) \asymp \begin{cases} \phi^{-1}(t^{-1})^{d/2} & t\phi(|x|^{-2}) > 1 \\ \frac{tH(|x|^{-2}) \vee e^{-a|x|^2 \phi^{-1}(t^{-1})}}{|x|^d} & t\phi(|x|^{-2}) \le 1\,, \end{cases}$$

where $a = a_L$ for the lower and $a = a_U$ for the upper bound.

- $\lambda_{L,\phi} = \lambda_{L,H} = \lambda_{U,H} = 0 \implies$ estimates are global in time and space



$$t\phi(|x|^{-2}) = 1$$

$$tH(|x|^{-2}) = e^{-a|x|^2\phi^{-1}(t^{-1})}$$

'Classical' case

If the scaling is strictly between 0 and 2 we get known estimate:

Corollary

If the Laplace exponent ϕ *satisfies* **(L)** *and* **(U)** *with* $\delta < 1$, *then there exists* $\kappa > 0$ *such that for* $0 < t < \phi(\lambda_L)^{-1}$ *and* $|x| < \kappa \lambda_U^{-1/2}$

$$p(t,x) \asymp \begin{cases} \phi^{-1}(t^{-1})^{d/2} & t\phi(|x|^{-2}) > 1 \\ \frac{t\phi(|x|^{-2})}{|x|^d} & t\phi(|x|^{-2}) \le 1 \end{cases}$$

$$\asymp \phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d}.$$

Second estimate follows from the following observation

$$t\phi(|x|^{-2}) \le 1 \iff \phi^{-1}(t^{-1})^{d/2} \ge |x|^{-d}.$$

'Dual' of geometric β-stable process

$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}$ for some $0 < \beta < 2$

'Dual' of geometric β-stable process

$$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})} \text{ for some } 0 < \beta < 2$$

$$\phi^{-1}(\lambda) \asymp \begin{cases} \lambda^{\frac{2}{2-\beta}} & 0 < \lambda < 2 \\ \lambda \log \lambda & \lambda \geq 2 \end{cases} \qquad H(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0 < \lambda < 2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2 \end{cases}$$

'Dual' of geometric β-stable process

$$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})} \text{ for some } 0 < \beta < 2$$

$$\phi^{-1}(\lambda) \asymp \begin{cases} \lambda^{\frac{2}{2-\beta}} & 0 < \lambda < 2 \\ \lambda \log \lambda & \lambda \geq 2 \end{cases} \qquad H(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0 < \lambda < 2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2 \end{cases}$$

- H satisfies **(L)** and **(U)** with $\lambda_L = \lambda_U = 0$ and $\delta < 2$

'Dual' of geometric β-stable process

$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}$ for some $0 < \beta < 2$

$$\phi^{-1}(\lambda) \asymp \begin{cases} \lambda^{\frac{2}{2-\beta}} & 0 < \lambda < 2 \\ \lambda \log \lambda & \lambda \geq 2 \end{cases} \qquad H(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0 < \lambda < 2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2 \end{cases}$$

- H satisfies **(L)** and **(U)** with $\lambda_L = \lambda_U = 0$ and $\delta < 2$
- ϕ and H are not comparable for large λ

'Dual' of geometric β-stable process

$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}$ for some $0 < \beta < 2$

$$\phi^{-1}(\lambda) \asymp \begin{cases} \lambda^{\frac{2}{2-\beta}} & 0 < \lambda < 2 \\ \lambda \log \lambda & \lambda \geq 2 \end{cases} \qquad H(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0 < \lambda < 2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2 \end{cases}$$

- H satisfies **(L)** and **(U)** with $\lambda_L = \lambda_U = 0$ and $\delta < 2$
- ϕ and H are not comparable for large λ
- off-diagonal estimate for 'small time and space':

$$p(t,x) \asymp \frac{t}{|x|^{d+2}(\log \frac{1}{|x|})^2} \vee \frac{e^{-a\frac{|x|^2}{t}\log\frac{1}{t}}}{|x|^d}$$

'Dual' of geometric β-stable process

$\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}$ for some $0 < \beta < 2$

$$\phi^{-1}(\lambda) \asymp \begin{cases} \lambda^{\frac{2}{2-\beta}} & 0 < \lambda < 2 \\ \lambda \log \lambda & \lambda \geq 2 \end{cases} \qquad H(\lambda) \asymp \begin{cases} \lambda^{1-\beta/2} & 0 < \lambda < 2 \\ \frac{\lambda}{(\log \lambda)^2} & \lambda \geq 2 \end{cases}$$

- H satisfies **(L)** and **(U)** with $\lambda_L = \lambda_U = 0$ and $\delta < 2$
- ϕ and H are not comparable for large λ
- off-diagonal estimate for 'small time and space':

$$p(t,x) \asymp \frac{t}{|x|^{d+2}(\log \frac{1}{|x|})^2} \vee \frac{e^{-a\frac{|x|^2}{t}\log\frac{1}{t}}}{|x|^d}$$

- Kaleta, Sztonyk '14

Theorem (On-diagonal upper bound, M. '15)

The on-diagonal upper bound

$$p(t,x) \leq C\phi^{-1}(t^{-1})^{d/2}\,, \qquad 0 < t < \phi(\lambda_L)^{-1}\,,\;\; x \in \mathbb{R}^d$$

holds if and only if ϕ *satisfies* **(L)** .

Theorem (On-diagonal upper bound, M. '15)

The on-diagonal upper bound

$$p(t,x) \le C\phi^{-1}(t^{-1})^{d/2}, \qquad 0 < t < \phi(\lambda_L)^{-1}, \;\; x \in \mathbb{R}^d$$

holds if and only if ϕ *satisfies* **(L)** .

Idea of proof.

Theorem (On-diagonal upper bound, M. '15)

The on-diagonal upper bound

$$p(t,x) \le C\phi^{-1}(t^{-1})^{d/2}, \qquad 0 < t < \phi(\lambda_L)^{-1}, \ x \in \mathbb{R}^d$$

holds if and only if ϕ *satisfies* **(L)** .

Idea of proof.

$$C\phi^{-1}(t^{-1})^{d/2} \ge p(t,0) \ge \int_{(0,2a\phi^{-1}(at^{-1})^{-1})} (4\pi s)^{-1}\mathbb{P}(S_t \in ds)$$
$$\ge (8\pi a)^{-d/2}\phi^{-1}(at^{-1})^{d/2}\mathbb{P}(S_t < 2a\phi^{-1}(at^{-1})^{-1})$$

Theorem (On-diagonal upper bound, M. '15)

The on-diagonal upper bound

$$p(t,x) \le C\phi^{-1}(t^{-1})^{d/2}, \qquad 0 < t < \phi(\lambda_L)^{-1}, \;\; x \in \mathbb{R}^d$$

holds if and only if ϕ *satisfies* **(L)** .

Idea of proof.

$$C\phi^{-1}(t^{-1})^{d/2} \ge p(t,0) \ge \int_{(0,2a\phi^{-1}(at^{-1})^{-1})} (4\pi s)^{-1}\mathbb{P}(S_t \in ds)$$
$$\ge (8\pi a)^{-d/2}\phi^{-1}(at^{-1})^{d/2}\mathbb{P}(S_t < 2a\phi^{-1}(at^{-1})^{-1})$$

$$\mathbb{P}(S_t < 2a\phi^{-1}(at^{-1})^{-1}) = 1 - \mathbb{P}(1 - e^{-\phi^{-1}(at^{-1})S_t} \ge 1 - e^{-2a})$$
$$\ge 1 - \frac{1 - e^{-t\phi(\phi^{-1}(at^{-1}))}}{1 - e^{-2a}} = \frac{e^{-a} - e^{-2a}}{1 - e^{-2a}}$$

Theorem (On-diagonal upper bound, M. '15)

The on-diagonal upper bound

$$p(t,x) \le C\phi^{-1}(t^{-1})^{d/2}, \quad 0 < t < \phi(\lambda_L)^{-1}, \ x \in \mathbb{R}^d$$

holds if and only if ϕ *satisfies* **(L)** .

Idea of proof.

$$C\phi^{-1}(t^{-1})^{d/2} \ge p(t,0) \ge \int_{(0,2a\phi^{-1}(at^{-1})^{-1})} (4\pi s)^{-1}\mathbb{P}(S_t \in ds)$$
$$\ge (8\pi a)^{-d/2}\phi^{-1}(at^{-1})^{d/2}\mathbb{P}(S_t < 2a\phi^{-1}(at^{-1})^{-1})$$

$$\mathbb{P}(S_t < 2a\phi^{-1}(at^{-1})^{-1}) = 1 - \mathbb{P}(1 - e^{-\phi^{-1}(at^{-1})S_t} \ge 1 - e^{-2a})$$
$$\ge 1 - \frac{1 - e^{-t\phi(\phi^{-1}(at^{-1}))}}{1 - e^{-2a}} = \frac{e^{-a} - e^{-2a}}{1 - e^{-2a}}$$

$$\Longrightarrow \quad \frac{\phi^{-1}(at^{-1})}{\phi^{-1}(t^{-1})} \le b \text{ for all } 0 < t < \phi(\lambda_L)^{-1} \Longrightarrow \textbf{(L)} \quad \square$$

Idea of the proof of the off-diagonal upper bound.

Idea of the proof of the off-diagonal upper bound.

$$\begin{aligned}\mathbb{P}_0(|X_t| \geq r) &= \int_{(0,\infty)} \mathbb{P}_0(|B_s|^2 \geq r^2)\mathbb{P}(S_t \in ds)\\ &= \int_{(0,\infty)} \mathbb{P}_0(\tfrac{|B_s|^2}{2s} \geq \tfrac{r^2}{2s})\mathbb{P}(S_t \in ds)\end{aligned}$$

Idea of the proof of the off-diagonal upper bound.

$$\begin{aligned}\mathbb{P}_0(|X_t| \geq r) &= \int_{(0,\infty)} \mathbb{P}_0(|B_s|^2 \geq r^2)\mathbb{P}(S_t \in ds) \\ &= \int_{(0,\infty)} \mathbb{P}_0\left(\frac{|B_s|^2}{2s} \geq \frac{r^2}{2s}\right)\mathbb{P}(S_t \in ds)\end{aligned}$$

Since $Y = \frac{|B_s|^2}{2s} \sim \chi^2(d)$ as a sum of d independent standard normal rv's (independent of S!) ,

Idea of the proof of the off-diagonal upper bound.

$$\begin{aligned}\mathbb{P}_0(|X_t| \geq r) &= \int_{(0,\infty)} \mathbb{P}_0(|B_s|^2 \geq r^2)\mathbb{P}(S_t \in ds)\\ &= \int_{(0,\infty)} \mathbb{P}_0(\tfrac{|B_s|^2}{2s} \geq \tfrac{r^2}{2s})\mathbb{P}(S_t \in ds)\end{aligned}$$

Since $Y = \frac{|B_s|^2}{2s} \sim \chi^2(d)$ as a sum of d independent standard normal rv's (independent of S!) ,

$$\begin{aligned}\mathbb{P}_0(|X_t| \geq r) &= \int_{(0,\infty)} \mathbb{P}_0(Y \geq \tfrac{r^2}{2s})\mathbb{P}(S_t \in ds)\\ &= \int_{(0,\infty)} \mathbb{P}(S_t \geq \tfrac{r^2}{2y})\mathbb{P}_0(Y \in dy)\,.\end{aligned}$$

Proposition (Tail estimate for subordinators, M. '15)

If $t\phi(r^{-1}) \leq \frac{1}{2}$, *then*

$$\mathbb{P}(S_t \geq r) \leq 2etH(r^{-1})\,.$$

Proposition (Tail estimate for subordinators, M. '15)

If $t\phi(r^{-1}) \le \frac{1}{2}$, *then*

$$\mathbb{P}(S_t \ge r) \le 2etH(r^{-1})\,.$$

$$\begin{aligned}
\mathbb{P}(|X_t| \ge r/2) &\le \int_{t\phi(2y(r/2)^{-2})\le\frac{1}{2}} 2etH(2y(r/2)^{-2})\mathbb{P}_0(Y \in dy) \\
&\quad + \int_{t\phi(2y(r/2)^{-2})\ge\frac{1}{2}} \mathbb{P}_0(Y \in dy) \\
&\le c_1 tH(r^{-2}) \int_{(0,\infty)} (1 + c_2 y^2)\mathbb{P}_0(Y \in dy) \\
&\quad + \mathbb{P}_0(Y \ge c_3 r^2 \phi^{-1}(\tfrac{t^{-1}}{2})) \\
&\le c_4 tH(r^{-2}) + c_5 e^{-a_U r^2 \phi^{-1}(t^{-1})}\,,
\end{aligned}$$

since $\mathbb{P}(Y \ge y) = 2^{-d/2}\Gamma(d/2)^{-1} \int_y^\infty s^{d/2-1} e^{-s/2}\, ds\,.$

It is left to notice that

$$c_6|x|^d p(t,x) \le \mathbb{P}_0(|x|/2 \le X_t \le |x|) \\ \le c_4 t H(r^{-2}) + c_5 e^{-a_U r^2 \phi^{-1}(t^{-1})}\,.$$

- For upper bounds, only **(L)** for ϕ is needed, since

$$\frac{H(\lambda x)}{H(\lambda)} \le x^2 \quad \text{for all} \quad \lambda > 0,\ x \ge 1$$

always holds.

Application to Green function (potential)

Application to Green function (potential)

Let X be transient, i.e.

$$\mathbb{P}_0(\lim_{t\to\infty} |X_t| = \infty) = 1\,.$$

Application to Green function (potential)

Let X be transient, i.e.

$$\mathbb{P}_0(\lim_{t\to\infty} |X_t| = \infty) = 1\,.$$

Then the following function is well-defined and finite

$$G(x,y) = G(y-x) = \int_0^\infty p(t,y-x)\,dt,\ \ x \neq y$$

and it is called the Green function or potential of X.

Application to Green function (potential)

Let X be transient, i.e.

$$\mathbb{P}_0(\lim_{t\to\infty} |X_t| = \infty) = 1\,.$$

Then the following function is well-defined and finite

$$G(x,y) = G(y-x) = \int_0^\infty p(t, y-x)\, dt,\ \ x \neq y$$

and it is called the Green function or potential of X .

Proposition

Assume that $\lambda/\phi(\lambda)$ is also a Bernstein function and that ϕ satisfies **(L)** *with $\lambda_L = 0$. Then*

$$G(x) \asymp \frac{1}{|x|^d \phi(|x|^{-2})},\ \ x \in \mathbb{R}^d, x \neq 0\ .$$

Idea of proof. Here we use:

Idea of proof. Here we use:

$$\frac{\phi^{-1}(\lambda x)}{\phi^{-1}(\lambda)} \geq x \quad \text{for all} \;\; x \geq 1 \;\; \text{and} \;\; \lambda > 0\,,$$

(since $\frac{\phi(\lambda x)}{\phi(\lambda)} \leq x$) .

Idea of proof. Here we use:

$$\frac{\phi^{-1}(\lambda x)}{\phi^{-1}(\lambda)} \geq x \quad \text{for all } \; x \geq 1 \; \text{ and } \; \lambda > 0 \,,$$

(since $\frac{\phi(\lambda x)}{\phi(\lambda)} \leq x$) .

$$G(x) = \int_0^{\phi(|x|^{-2})^{-1}} p(t,x)\,dt + \int_{\phi(|x|^{-2})^{-1}}^{\infty} p(t,x)\,dt = I_1 + I_2 \,.$$

Idea of proof. Here we use:

$$\frac{\phi^{-1}(\lambda x)}{\phi^{-1}(\lambda)} \geq x \quad \text{for all } \; x \geq 1 \; \text{ and } \; \lambda > 0\,,$$

(since $\frac{\phi(\lambda x)}{\phi(\lambda)} \leq x$) .

$$G(x) = \int_0^{\phi(|x|^{-2})^{-1}} p(t,x)\,dt + \int_{\phi(|x|^{-2})^{-1}}^{\infty} p(t,x)\,dt = I_1 + I_2\,.$$

$$\begin{aligned} G(x) \geq I_2 &\geq c_1 \int_{\phi(|x|^{-2})^{-1}}^{2\phi(|x|^{-2})^{-1}} \phi^{-1}(t^{-1})^{d/2}\,dt \\ &\geq c_1 \phi^{-1}\left(\frac{\phi(|x|^{-2})}{2}\right)^{d/2} \frac{1}{\phi(|x|^{-2})} \geq \frac{c_1}{|x|^d \phi(|x|^{-2})} \end{aligned}$$

We only prove bound for I_2:

$$
\begin{aligned}
I_2 &\le c_2|x|^{-d}\int_0^{\phi(|x|^{-2})^{-1}} \left[tH(|x|^{-2}) + e^{-a_U|x|^2\phi^{-1}(t^{-1})}\right] dt \\
&\le c_2|x|^{-d}\frac{H(|x|^{-2})}{2\phi(|x|^{-2})^2} + c_3|x|^{-d}\int_0^{\phi(|x|^{-2})^{-1}} \left[|x|^2\phi^{-1}(t^{-1})\right]^{-1} dt \\
&\le \frac{c_4}{|x|^d\phi(|x|^{-2})} + c_5\int_0^{\phi(|x|^{-2})^{-1}} \frac{\phi^{-1}(\phi(|x|^{-2}))}{\phi^{-1}\left(\frac{\phi(|x|^{-2})}{t\phi(|x|^{-2})}\right)}\, dt \\
&\le \frac{c_4}{|x|^d\phi(|x|^{-2})} + c_6\int_0^{\phi(|x|^{-2})^{-1}} t\phi(|x|^{-2})\, dt \le \frac{c_7}{|x|^d\phi(|x|^{-2})}\,.
\end{aligned}
$$

□

If $d \geq 3$ we obtain for 'dual' variance gamma process, i.e. for $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1$, If $d \geq 3$ we obtain for 'dual' variance gamma process, i.e. for $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1$, the following estimates

$$G(x) \asymp \begin{cases} |x|^{2-d} \log \frac{1}{|x|} & |x| < \frac{1}{2} \\ |x|^{2-d} & |x| \geq \frac{1}{2}\,, \end{cases}$$

If $d \geq 3$ we obtain for 'dual' variance gamma process, i.e. for $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1$, the following estimates

$$G(x) \asymp \begin{cases} |x|^{2-d} \log \frac{1}{|x|} & |x| < \frac{1}{2} \\ |x|^{2-d} & |x| \geq \frac{1}{2} \end{cases},$$

and this is very close to the Green function of Brownian motion (Newtonian potential)

$$G^{(2)}(x) = |x|^{2-d}.$$

Thank

you

for

your

attention!