On the asymptotic behavior of the density of the supremum of Lévy processes

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L. Chaumont, JM, 2015
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Notation: part I

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$$\mathbf{E}e^{i\xi X_t} = e^{-t\Psi(\xi)}, \quad t\geq 0, \xi\in\mathbb{R}.$$

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- Supremum functional

$$\overline{X}_t = \sup_{0\leq s\leq t} X_s$$

and its density function (if it exists)

$$f_t(x) = \frac{\mathbf{P}(\overline{X}_t \in dx)}{dx}, \quad x > 0, t > 0.$$

- We denote by $\kappa(z,\xi)$ the Laplace exponent of the bivariate subordinator (τ_s, H_s).

$$\kappa(z,\xi) = \exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-zt-\xi x}) t^{-1} \mathbf{P}(X_t \in dx)\right)$$

and

$$\kappa(z,0) = \mathtt{d}z + \int_{(0,\infty)} (1 - e^{-zx})\pi(dx)$$

where $\mathtt{d}$ and π are a drift and a Levy measure of ladder-time process τ_s respectively.

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- Renewal function of the process H_s

$$h(x) = \int_0^\infty \mathbf{P}(H_s < x) ds$$

  and its derivative

$$h'(x) = \int_0^\infty \mathbf{P}(H_s \in dx) ds/dx.$$

Motivations

- L. Chaumont, 2013
  On the law of the supremum of Lévy processes
  Annals Probab. vol. 41, no. 3A (2013)

- M. Kwaśnicki, JM, M. Ryznar
  Suprema of Lévy processes
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- necessary and sufficient conditions for f_t to exists
- the formula for the distribution of $(X_t, \overline{X}_t, g_t)$ in terms of the entrance law $q_t(dx)$ (tool to study properties of $f_t(x)$)

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- the estimates

$$\mathbf{P}(\overline{X}_t < x) \approx \min(1, \kappa(1/t, 0)h(x)), \quad x, t > 0,$$

whenever $\kappa(z, 0)$ has some regularity property at zero and infinity.

Questions

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is it true that

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- What is the asymptotic behaviour of the density $f_t(x)$, when x tends to 0 (for fixed t).
- What is the asymptotic behaviour of the density $f_t(x)$, when t tends to ∞ (for fixed x).
- Is f_t continuous?

Assumptions

(H2) $(X, \mathbf{P})$ is not a compound Poisson process and for all $c \geq 0$, the process $((|X_t - ct|, t \geq 0), \mathbf{P})$ is not a subordinator.

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(H1) The transition semigroup of $(X, \mathbf{P})$ is absolutely continuous and there is a version of its densities, denoted by $x \mapsto p_t(x)$, $x \in \mathbf{R}$, which are bounded for all $t > 0$.

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Moreover, **(H1)** implies that $f_t(x)$ exists on $(0, \infty)$ (in same cases $\mathbf{P}(\overline{X}_t \in dx)$ might have an atom at 0).

Continuity of f_t

Under **(H1)**, continuity of f_t is equivalent to the continuity of h' in the following sense:

Theorem (L. Chaumont, JM, 2015)

The following conditions are equivalent:

- f_t *is continuous at* $x_0 > 0$ *for every* $t > 0$,
- f_t *is continuous at* $x_0 > 0$ *for some* $t > 0$,
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- In many (many) cases h' is continuous on $(0, \infty)$.
- However sometimes (for example, when X has no negative jumps, bounded variations and a Lévy measure which admits atoms) h' is not continuous.

Asymptotic behaviour when $x \to 0^+$

Recall that the ladder time process τ_t is (possibly killed) subordinator with Lévy measure π (with killing rate $\pi(\infty)$).

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The density of the law of the past supremum of $(X, \boldsymbol{P})$ fulfills the following asymptotic behaviour,

$$\lim_{x \to 0^+} \frac{f_t(x)}{h'(x)} = \pi(t, \infty)$$

uniformly on $[t_0, \infty)$ for every fixed $t_0 > 0$.

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- If $z \to \kappa(z, 0) \in R_\rho(0)$, $\rho \in [0, 1)$ then $\kappa(1/t, 0) \sim \Gamma(1-\rho)\pi(t, \infty)$ as $t \to \infty$.

Asymptotic behaviour when $t \to \infty$

Theorem (L. Chaumont, JM 2015)

If we additionally assume that $\pi(t,\infty) \in R_{-\rho}(\infty)$, $\rho \in (0,1)$ then

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uniformly in x on every compact subset of $(0,\infty)$.

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Note that the following conditions are equivalent

- $\pi(t,\infty) \in R_{-\rho}(\infty)$,
- $z \to \kappa(z,0) \in R_{\rho}(0)$,
- $\lim_{t\to\infty} \mathbf{P}(X_t \geq 0) = \rho$.

Estimates of f_t

Theorem (L. Chaumont, JM 2015)

For every fixed $x_0, t_0 > 0$ *there exist constants* $c_1, c_2 > 0$ *such that*

$$c_1\,\pi(t,\infty) \le \frac{f_t(x)}{h'(x)} \le c_2 \int_0^t \pi(s,\infty)ds, \quad x \le x_0, t \ge t_0$$

and if additionally $\pi(t,\infty) \in R_{-\rho}(\infty)$, $\rho \in (0,1)$ *then*

$$f_t(x) \overset{x_0,t_0}{\approx} \pi(t,\infty)h'(x) \approx \kappa(1/t,0)h'(x)$$

for $x \le x_0$ *and* $t \ge t_0$.

Notation: part II

- $\overline{X}_t - X_t$ - the process reflected in its supremum (strong Markov process)
- n - the Ito measure of the excursions away from 0 for the reflected process
- $q_t(dx)$ the entrance laws of the reflected excursions at the maximum, i.e.

$$\int_{[0,\infty)} f(x)q_t(dx) = n(f(X_t), t < \zeta),$$

  for any positive Borel function (ζ - lifetime).
- in our setting $q_t(dx) = q_t(x)dx$

Analogously, we define $q_t^*(dx)$ and $q_t^*(x)$.

The law of the triple $(X_t, \overline{X}_t, g_t)$ is given in terms of q_t and q_t^*:

Theorem (L. Chaumont, AoP 2013)

$$\boldsymbol{P}(\overline{X}_t \in dx, \overline{X}_t - X_t \in dy, g_t \in ds) = q_t^*(dx) q_{t-s}(dy) \boldsymbol{1}_{[0,t]}(ds) + \mathtt{d}\delta_t(ds)\delta_0(dy) q_t^*(dx) + \mathtt{d}^*\delta_t(ds)\delta_0(dx) q_t(dy)$$

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Moreover we have

$$q_t(\mathbf{R}_+) = \int_0^\infty q_t(x)dx = n(t < \zeta) = \pi(t,\infty).$$

$$\int_0^\infty q_t^*(x)dt = h'(x).$$

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$$\int_0^\infty q_t^*(x)dt = h'(x).$$

Thus, by simple integration, we get

$$f_t(x) = \int_0^t q_s^*(x) n(t-s < \zeta) ds + \mathtt{d} q_t^*(x), \quad x > 0.$$

Step I: asymptotics of $q_t(x)$ when $x \to 0^+$

Let us denote by $q_t(x, dy)$ the semigroup of the process X killed when it enters in the negative half-line.

If **(H1)** holds then $q_t(x, dy) = q_t(x, y)dy$.

Proposition (Uribe 2014, L. Chaumont, JM 2015)

We have

$$q_t^*(x) = \lim_{y \to 0^+} \frac{q_t(x, y)}{h^*(y)}, \quad \lim_{x \to 0^+} \frac{q_t^*(x)}{h(x)} = \frac{p_t(0)}{t}.$$

Recall

$$f_t(x) = \int_0^t q_s^*(x) n(t-s<\zeta) ds + \mathtt{d} q_t^*(x), \quad x>0.$$

Thus

$$\frac{f_t(x)}{h'(x)} = \int_0^\delta n(t-s<\zeta)\frac{q_s^*(x)}{h'(x)} ds + \frac{h(x)}{h'(x)}\int_\delta^t n(t-s<\zeta)\frac{q_s^*(x)}{h(x)} ds + \mathtt{d}\frac{h(x)}{h'(x)}\frac{q_t^*(x)}{h(x)}$$

and we need to show that

$$\lim_{x\to 0^+} \frac{h(x)}{h'(x)} = 0$$

and find some upper-bounds for

$$\frac{q_t^*(x)}{h(x)}.$$

Step II: $\lim_{x\to 0^+} h(x)/h'(x) = 0$

First we show that

Proposition (L. Chaumont, JM 2015)

If p_t is bounded for some $t > 0$ then

$$\int^{\infty} \frac{p_t(0)}{t} dt < \infty.$$

If p_t is bounded for every $t > 0$ then

$$\int_{0^+} \frac{p_t(0)}{t} dt = \infty.$$

Thus, by Fatou Lemma and $h'(x) = \int_0^\infty q_t^*(x)dt$, we get

$$\liminf_{x\to 0^+} \frac{h'(x)}{h(x)} \geq \int_0^\infty \liminf_{x\to 0^+} \frac{q_t^*(x)}{h(x)} dt = \int_0^\infty \frac{p_t(0)}{t} dt = \infty.$$

Step III: upper-bounds for $q_t^*(x)/h(x)$

Proposition (L. Chaumont, JM 2015)

There exists $c > 0$ such that

$$\frac{q_t^*(x)}{h(x)} \leq c \frac{p_{t/6}^S(0)}{t}, \quad x > 0, t > 0.$$

Here

$$p_t^S = p_t * \check{p}_t$$

is the transition density function of the symmetrization of X.

Obviously, under **(H1)**, p_t^S is also bounded for every $t > 0$.

Consequently, the above-given upper-bounds are integrable at infinity (in t).

Step III: upper-bounds for $q_t^*(x)/h(x)$

Since $e^{-t\Psi(\xi)}$ is integrable

$$q_t^*(x,y) \le p_t(y-x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{i\xi(x-y)}e^{-t\Psi(\xi)}d\xi \le p^S_{t/2}(0).$$

Moreover (M. Kwasnicki, JM, M. Ryznar, AoP 2013)

$$\begin{aligned}\int_0^\infty q_t(x,y)dy &= \mathbf{P}(\overline{X}_t < x) \le \frac{e}{e-1}\kappa(1/t,0)h(x)\\ \int_0^\infty q_t^*(x,y)dy &\le \frac{e}{e-1}\kappa^*(1/t,0)h^*(x).\end{aligned}$$

Thus, by Chapmann-Kolmogorov equation

$$\begin{aligned}
q^*_{3t}(x,y) &= \int_0^\infty \int_0^\infty q^*_t(x,z)q^*_t(z,w)q^*_t(w,y)dzdw \\
&\leq p^S_{t/2}(0) \int_0^\infty q^*_t(x,z)dz \int_0^\infty q^*_t(w,y)\,dw \\
&= p^S_{t/2}(0) \int_0^\infty q^*_t(x,z)dz \int_0^\infty q_t(y,w)\,dw \\
&\leq \left(\frac{e}{e-1}\right)^2 p^S_{t/2}(0)h^*(x)h(y)\kappa(1/t,0)\kappa^*(1/t,0) \\
&= \left(\frac{e}{e-1}\right)^2 \frac{p^S_{t/2}(0)}{t} h^*(x)h(y).
\end{aligned}$$

Consequently, for every $y > 0$ and $t > 0$ we have

$$\frac{q^*_t(y)}{h(y)} = \lim_{x\to 0^+} \frac{q^*_t(x,y)}{h^*(x)h(y)} \leq 3\left(\frac{e}{e-1}\right)^2 \frac{p^S_{t/6}(0)}{t}.$$

Writing once again

$$\begin{aligned}\frac{f_t(x)}{h'(x)} &= \int_0^\delta n(t-s<\zeta)\frac{q_s^*(x)}{h'(x)}ds + \frac{h(x)}{h'(x)}\int_\delta^t n(t-s<\zeta)\frac{q_s^*(x)}{h(x)}ds \\ &\quad + \mathtt{d}\frac{h(x)}{h'(x)}\frac{q_t^*(x)}{h(x)}\end{aligned}$$

we can see that

$$\begin{aligned}\limsup_{x\to 0^+}\frac{f_t(x)}{h'(x)} &\leq (n(t<\zeta)-\varepsilon)\limsup_{x\to 0^+}\frac{1}{h'(x)}\int_0^\delta q_t^*(x)dx \\ &\leq n(t<\zeta)-\varepsilon\end{aligned}$$

To deal with lower bounds, we use monotonicity of $n(t < \zeta)$ and we write

$$\begin{aligned} f_t(x) &= \int_0^t q_s^*(x) n(t-s<\zeta) ds + \mathtt{d} q_t^*(x) \\ &\geq n(t<\zeta) \int_0^t q_s^*(x) ds \end{aligned}$$

and consequently

$$\begin{aligned} \liminf_{x \to 0^+} \frac{f_t(x)}{h'(x)} &\geq n(t<\zeta) - \limsup_{x\to\infty} \frac{h(x)}{h'(x)} \int_t^\infty \frac{q_s^*(x)}{h'(x)} ds \\ &\geq n(t<\zeta) \end{aligned}$$

Thank you very much.