On the asymptotic behavior of the density of the supremum of Lévy processes

Jacek Małecki

Departament of Mathematics
Wrocław University of Technology

Probability and Analysis
Będlewo, May 5, 2015

L. Chaumont, JM, 2015
On the asymptotic behavior of the density of the supremum of Lévy processes
Notation: part I

Let $(X_t)_{t \geq 0}$ be a one-dimensional Lévy process, i.e. a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories.

$\Psi(\xi) - \text{Lévy-Khintchin exponent of } X_t$, i.e. we have $E e^{i \xi X_t} = e^{-t \Psi(\xi)}$, $t \geq 0$, $\xi \in \mathbb{R}$.

Supremum functional $X_t = \sup_{0 \leq s \leq t} X_s$ and its density function (if it exists) $f_t(x) = P(X_t \in dx)$, $x > 0$, $t > 0$. 
Notation: part I

- Let \((X_t)_{t \geq 0}\) be a one-dimensional Lévy process, i.e. a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories.

- \(\Psi(\xi)\) - Lévy-Khintchin exponent of \(X_t\), i.e. we have

\[
E e^{i \xi X_t} = e^{-t \Psi(\xi)}, \quad t \geq 0, \xi \in \mathbb{R}.
\]
Notation: part I

- Let \((X_t)_{t \geq 0}\) be a one-dimensional Lévy process, i.e. a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories.
- \(\Psi(\xi)\) - Lévy-Khintchin exponent of \(X_t\), i.e. we have
  \[
  \mathbb{E}e^{i\xi X_t} = e^{-t\Psi(\xi)}, \quad t \geq 0, \xi \in \mathbb{R}.
  \]
- Supremum functional
  \[
  \overline{X}_t = \sup_{0 \leq s \leq t} X_s
  \]
  and its density function (if it exists)
  \[
  f_t(x) = \frac{\mathbb{P}(\overline{X}_t \in dx)}{dx}, \quad x > 0, t > 0.
  \]
We denote by $\kappa(z, \xi)$ the Laplace exponent of the bivariate subordinator $(\tau_s, H_s)$.

\[ \kappa(z, \xi) = \exp \left( \int_0^{\infty} \int_{[0, \infty)} (e^{-t} - e^{-zt-\xi x}) t^{-1} P(X_t \in dx) \right) \]

and

\[ \kappa(z, 0) = dz + \int_{(0, \infty)} (1 - e^{-zx}) \pi(dx) \]

where $d$ and $\pi$ are a drift and a Levy measure of ladder-time process $\tau_s$ respectively.
We denote by $\kappa(z, \xi)$ the Laplace exponent of the bivariate subordinator $(\tau_s, H_s)$.

$$\kappa(z, \xi) = \exp \left( \int_0^\infty \int_{[0,\infty)} \left( e^{-t} - e^{-zt-\xi x} \right) t^{-1} P(X_t \in dx) \right)$$

and

$$\kappa(z, 0) = dz + \int_{(0,\infty)} (1 - e^{-zx}) \pi(dx)$$

where $d$ and $\pi$ are a drift and a Levy measure of ladder-time process $\tau_s$ respectively.

Renewal function of the process $H_s$

$$h(x) = \int_0^\infty P(H_s < x) ds$$

and its derivative

$$h'(x) = \int_0^\infty P(H_s \in dx) ds / dx.$$
Motivations

L. Chaumont, 2013

*On the law of the supremum of Lévy processes*
Annals Probab. vol. 41, no. 3A (2013)

M. Kwaśnicki, JM, M. Ryznar

*Suprema of Lévy processes*
Annals Probab. vol. 41, no. 3B (2013)
Motivations

L. Chaumont, 2013
*On the law of the supremum of Lévy processes*
Annals Probab. vol. 41, no. 3A (2013)

- necessary and sufficient conditions for $f_t$ to exists

M. Kwaśnicki, JM, M. Ryznar
*Suprema of Lévy processes*
Annals Probab. vol. 41, no. 3B (2013)
Motivations

L. Chaumont, 2013
On the law of the supremum of Lévy processes
Annals Probab. vol. 41, no. 3A (2013)

- necessary and sufficient conditions for $f_t$ to exist
- the formula for the distribution of $(X_t, \overline{X}_t, g_t)$ in terms of the entrance law $q_t(dx)$ (tool to study properties of $f_t(x)$)

M. Kwaśnicki, JM, M. Ryznar
Suprema of Lévy processes
Annals Probab. vol. 41, no. 3B (2013)
Motivations

L. Chaumont, 2013

*On the law of the supremum of Lévy processes*
Annals Probab. vol. 41, no. 3A (2013)

- necessary and sufficient conditions for \( f_t \) to exists
- the formula for the distribution of \((X_t, X_t, g_t)\) in terms of the entrance law \( q_t(dx) \) (tool to study properties of \( f_t(x) \))

M. Kwaśnicki, JM, M. Ryznar

*Suprema of Lévy processes*
Annals Probab. vol. 41, no. 3B (2013)

- the estimates

\[
P(\overline{X}_t < x) \approx \min(1, \kappa(1/t, 0)h(x)), \quad x, t > 0,
\]

whenever \( \kappa(z, 0) \) has some regularity property at zero and infinity.
Questions

Since for $x$ small or $t$ large we have

$$P(\bar{X}_t < x) \approx \kappa(1/t, 0)h(x)$$

What is the asymptotic behaviour of the density $f_t(x)$, when $x$ tends to 0 (for fixed $t$).

What is the asymptotic behaviour of the density $f_t(x)$, when $t$ tends to $\infty$ (for fixed $x$).

Is $f_t(x)$ continuous?
Questions

Since for $x$ small or $t$ large we have

$$P(X_t < x) \approx \kappa(1/t, 0) h(x) \quad \text{and} \quad f_t(x) = \frac{d}{dx} P(X_t < x),$$

is it true that

$$f_t(x) \approx \kappa(1/t, 0) h'(x) \quad \text{for} \ x \ \text{small and} \ t \ \text{large}?$$
Questions

- Since for $x$ small or $t$ large we have

\[ P(X_t < x) \approx \kappa(1/t, 0)h(x) \quad \text{and} \quad f_t(x) = \frac{d}{dx}P(X_t < x), \]

is it true that

\[ f_t(x) \approx \kappa(1/t, 0)h'(x) \quad \text{for } x \text{ small and } t \text{ large?} \]

- What is the asymptotic behaviour of the density $f_t(x)$, when $x$ tends to 0 (for fixed $t$)?

- What is the asymptotic behaviour of the density $f_t(x)$, when $t$ tends to $\infty$ (for fixed $x$).
Questions

Since for $x$ small or $t$ large we have

$$P(\overline{X}_t < x) \approx \kappa(1/t, 0)h(x) \quad \text{and} \quad f_t(x) = \frac{d}{dx}P(\overline{X}_t < x),$$

is it true that

$$f_t(x) \approx \kappa(1/t, 0)h'(x) \quad \text{for } x \text{ small and } t \text{ large?}$$

What is the asymptotic behaviour of the density $f_t(x)$, when $x$ tends to 0 (for fixed $t$).

What is the asymptotic behaviour of the density $f_t(x)$, when $t$ tends to $\infty$ (for fixed $x$).

Is $f_t$ continuous?
Assumptions

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\),
the process \((|X_t - ct|, t \geq 0), P\) is not a subordinator.
Assumptions

(H1) The transition semigroup of \((X, P)\) is absolutely continuous and there is a version of its densities, denoted by \(x \mapsto p_t(x)\), \(x \in \mathbb{R}\), which are bounded for all \(t > 0\).

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\), the process \((|X_t - ct|, t \geq 0), P)\) is not a subordinator.
Assumptions

(H1) The transition semigroup of \((X, P)\) is absolutely continuous and there is a version of its densities, denoted by \(x \mapsto p_t(x)\), \(x \in \mathbb{R}\), which are bounded for all \(t > 0\).

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\), the process \(((|X_t - ct|, t \geq 0), P)\) is not a subordinator.

Note that

\[
(H1) \implies p_t \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \text{ for every } t > 0
\]
Assumptions

(H1) The transition semigroup of \((X, P)\) is absolutely continuous and there is a version of its densities, denoted by \(x \mapsto p_t(x)\), \(x \in \mathbb{R}\), which are bounded for all \(t > 0\).

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\), the process \(((|X_t - ct|, t \geq 0), P)\) is not a subordinator.

Note that

\[(H1) \quad \Rightarrow \quad p_t \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \text{ for every } t > 0
\]
\[
\Rightarrow \quad e^{-t\psi(\xi)} \in L^2(\mathbb{R}) \text{ for every } t > 0
\]
Assumptions

(H1) The transition semigroup of \((X, P)\) is absolutely continuous and there is a version of its densities, denoted by \(x \mapsto p_t(x)\), \(x \in \mathbb{R}\), which are bounded for all \(t > 0\).

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\), the process \(((|X_t - ct|, t \geq 0), P)\) is not a subordinator.

Note that

\[(H1) \implies p_t \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \text{ for every } t > 0\]
\[\implies e^{-t\psi(\xi)} \in L^2(\mathbb{R}) \text{ for every } t > 0\]
\[\implies e^{-t\psi(\xi)} \in L^1(\mathbb{R}) \text{ for every } t > 0\]
Assumptions

(H1) The transition semigroup of \((X, P)\) is absolutely continuous and there is a version of its densities, denoted by \(x \mapsto p_t(x), x \in \mathbb{R}\), which are bounded for all \(t > 0\).

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\), the process \(((|X_t - ct|, t \geq 0), P)\) is not a subordinator.

Note that

\[(H1) \implies p_t \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \text{ for every } t > 0\]
\[\implies e^{-t\psi(\xi)} \in L^2(\mathbb{R}) \text{ for every } t > 0\]
\[\implies e^{-t\psi(\xi)} \in L^1(\mathbb{R}) \text{ for every } t > 0\]
\[\implies p_t \in C_0(\mathbb{R}) \text{ for every } t > 0 \implies (H1)\]
Assumptions

(H1) The transition semigroup of \((X, P)\) is absolutely continuous and there is a version of its densities, denoted by \(x \mapsto p_t(x), x \in \mathbb{R}\), which are bounded for all \(t > 0\).

(H2) \((X, P)\) is not a compound Poisson process and for all \(c \geq 0\), the process \(((|X_t - ct|, t \geq 0), P)\) is not a subordinator.

Note that

\[(H1) \implies p_t \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \text{ for every } t > 0 \]
\[\implies e^{-t\psi(\xi)} \in L^2(\mathbb{R}) \text{ for every } t > 0 \]
\[\implies e^{-t\psi(\xi)} \in L^1(\mathbb{R}) \text{ for every } t > 0 \]
\[\implies p_t \in C_0(\mathbb{R}) \text{ for every } t > 0 \implies (H1)\]

Moreover, \((H1)\) implies that \(f_t(x)\) exists on \((0, \infty)\) (in some cases \(P(\overline{X}_t \in dx)\) might have an atom at 0).
Continuity of $f_t$

Under (H1), continuity of $f_t$ is equivalent to the continuity of $h'$ in the following sense:

**Theorem (L. Chaumont, JM, 2015)**

The following conditions are equivalent:

- $f_t$ is continuous at $x_0 > 0$ for every $t > 0$,
- $f_t$ is continuous at $x_0 > 0$ for some $t > 0$,
- $h'$ is continuous at $x_0 > 0$
Continuity of $f_t$

Under (H1), continuity of $f_t$ is equivalent to the continuity of $h'$ in the following sense:

**Theorem (L. Chaumont, JM, 2015)**

*The following conditions are equivalent:*

- $f_t$ is continuous at $x_0 > 0$ for every $t > 0$,
- $f_t$ is continuous at $x_0 > 0$ for some $t > 0$,
- $h'$ is continuous at $x_0 > 0$

- In many (many) cases $h'$ is continuous on $(0, \infty)$.
- However sometimes (for example, when $X$ has no negative jumps, bounded variations and a Lévy measure which admits atoms) $h'$ is not continuous.
Asymptotic behaviour when $x \to 0^+$

Recall that the ladder time process $\tau_t$ is (possibly killed) subordinator with Lévy measure $\pi$ (with killing rate $\pi(\infty)$).
Asymptotic behaviour when $x \to 0^+$

Recall that the ladder time process $\tau_t$ is (possibly killed) subordinator with Lévy measure $\pi$ (with killing rate $\pi(\infty)$).

**Theorem (L. Chaumont, JM 2015)**

The density of the law of the past supremum of $(X, P)$ fulfills the following asymptotic behaviour,

$$
\lim_{x \to 0^+} \frac{f_t(x)}{h'(x)} = \pi(t, \infty)
$$

uniformly on $[t_0, \infty)$ for every fixed $t_0 > 0$. 

Asymptotic behaviour when \( x \to 0^+ \)

Recall that the ladder time process \( \tau_t \) is (possibly killed) subordinator with Lévy measure \( \pi \) (with killing rate \( \pi(\infty) \)).

**Theorem (L. Chaumont, JM 2015)**

The density of the law of the past supremum of \((X, P)\) fulfills the following asymptotic behaviour,

\[
\lim_{x \to 0^+} \frac{f_t(x)}{h'(x)} = \pi(t, \infty)
\]

uniformly on \([t_0, \infty)\) for every fixed \(t_0 > 0\).

- If \( z \to \kappa(z, 0) \in R_\rho(0), \ \rho \in [0, 1) \) then
  \[
  \kappa(1/t, 0) \sim \Gamma(1 - \rho)\pi(t, \infty)
  \]  as \( t \to \infty \).
Asymptotic behaviour when \( t \to \infty \)

**Theorem (L. Chaumont, JM 2015)**

If we additionally assume that \( \pi(t, \infty) \in R_{-\rho}(\infty), \rho \in (0, 1) \) then

\[
\lim_{t \to \infty} \frac{f_t(x)}{\pi(t, \infty)} = h'(x)
\]

uniformly in \( x \) on every compact subset of \((0, \infty)\).
Asymptotic behaviour when $t \to \infty$

**Theorem (L. Chaumont, JM 2015)**

If we additionally assume that $\pi(t, \infty) \in R_{-\rho}(\infty)$, $\rho \in (0, 1)$ then

$$\lim_{t \to \infty} \frac{f_t(x)}{\pi(t, \infty)} = h'(x)$$

uniformly in $x$ on every compact subset of $(0, \infty)$.

Note that the following conditions are equivalent

- $\pi(t, \infty) \in R_{-\rho}(\infty)$,
- $z \to \kappa(z, 0) \in R_\rho(0)$,
- $\lim_{t \to \infty} P(X_t \geq 0) = \rho$. 
Estimates of $f_t$

Theorem (L. Chaumont, JM 2015)

For every fixed $x_0, t_0 > 0$ there exist constants $c_1, c_2 > 0$ such that

$$c_1 \pi(t, \infty) \leq \frac{f_t(x)}{h'(x)} \leq c_2 \int_0^t \pi(s, \infty) ds, \quad x \leq x_0, t \geq t_0$$

and if additionally $\pi(t, \infty) \in R_{-\rho}(\infty), \rho \in (0, 1)$ then

$$f_t(x) \overset{x_0, t_0}{\approx} \pi(t, \infty) h'(x) \approx \kappa(1/t, 0) h'(x)$$

for $x \leq x_0$ and $t \geq t_0$. 
Notation: part II

- $\bar{X}_t - X_t$ - the process reflected in its supremum (strong Markov process)
- $n$ - the Lto measure of the excursions away from 0 for the reflected process
- $q_t(dx)$ the entrance laws of the reflected excursions at the maximum, i.e.

$$
\int_{[0,\infty)} f(x)q_t(dx) = n(f(X_t), t < \zeta),
$$

for any positive Borel function ($\zeta$ - lifetime).

- in our setting $q_t(dx) = q_t(x)dx$

Analogously, we define $q_t^{*}(dx)$ and $q_t^{*}(x)$.
The law of the triple \((X_t, \overline{X}_t, g_t)\) is given in terms of \(q_t\) and \(q^*_t\):

**Theorem (L. Chaumont, AoP 2013)**

\[
P(\overline{X}_t \in dx, \overline{X}_t - X_t \in dy, g_t \in ds) = q^*_t(dx)q_{t-s}(dy)1_{[0,t]}(ds)
+ \delta_t(ds)\delta_0(dy)q^*_t(dx) + \delta^*_t(ds)\delta_0(dx)q_t(dy)
\]
The law of the triple \((X_t, \overline{X}_t, g_t)\) is given in terms of \(q_t\) and \(q_t^*\):

**Theorem (L. Chaumont, AoP 2013)**

\[
P(\overline{X}_t \in dx, \overline{X}_t - X_t \in dy, g_t \in ds) = q_t^*(dx)q_{t-s}(dy)\mathbf{1}_{[0,t]}(ds) \\
+ \delta_t(ds)\delta_0(dy)q_t^*(dx) + \delta_t(ds)\delta_0(dx)q_t(dy)
\]

Moreover we have

\[
q_t(\mathbb{R}^+) = \int_0^{\infty} q_t(x)dx = n(t < \zeta) = \pi(t, \infty).
\]

\[
\int_0^{\infty} q_t^*(x)dt = h'(x).
\]
The law of the triple \((X_t, \overline{X}_t, g_t)\) is given in terms of \(q_t\) and \(q^*_t\):

**Theorem (L. Chaumont, AoP 2013)**

\[
P(\overline{X}_t \in dx, \overline{X}_t - X_t \in dy, g_t \in ds) = q^*_t(dx)q_{t-s}(dy)1_{[0,t]}(ds) \\
+ \delta_t(ds)\delta_0(dy)q^*_t(dx) + \delta^*_{t}(ds)\delta_0(dx)q_t(dy)
\]

Moreover we have

\[
q_t(\mathbb{R}^+) = \int_0^\infty q_t(x)dx = n(t < \zeta) = \pi(t, \infty).
\]

\[
\int_0^\infty q^*_t(x)dt = h'(x).
\]

Thus, by simple integration, we get

\[
f_t(x) = \int_0^t q^*_s(x)n(t - s < \zeta)ds + \delta q^*_t(x), \quad x > 0.
\]
Step 1: asymptotics of $q_t(x)$ when $x \to 0^+$

Let us denote by $q_t(x, dy)$ the semigroup of the process $X$ killed when it enters in the negative half-line.

If (H1) holds then $q_t(x, dy) = q_t(x, y) dy$.

Proposition (Uribe 2014, L. Chaumont, JM 2015)

We have

$$q^*_t(x) = \lim_{y \to 0^+} \frac{q_t(x, y)}{h^*(y)}, \quad \lim_{x \to 0^+} \frac{q^*_t(x)}{h(x)} = \frac{p_t(0)}{t}.$$
Recall

\[ f_t(x) = \int_0^t q_s^*(x) n(t - s < \zeta) \, ds + dq_t^*(x), \quad x > 0. \]

Thus

\[
\frac{f_t(x)}{h'(x)} = \int_0^\delta n(t - s < \zeta) \frac{q_s^*(x)}{h'(x)} \, ds + \frac{h(x)}{h'(x)} \int_\delta^t n(t - s < \zeta) \frac{q_s^*(x)}{h(x)} \, ds
\]

\[
+ dq_t^* \frac{h(x)}{h'(x)} \frac{h(x)}{h(x)}
\]

and we need to show that

\[
\lim_{x \to 0^+} \frac{h(x)}{h'(x)} = 0
\]

and find some upper-bounds for

\[
\frac{q_t^*(x)}{h(x)}.
\]
Step II: \( \lim_{x \to 0^+} \frac{h(x)}{h'(x)} = 0 \)

First we show that

**Proposition (L. Chaumont, JM 2015)**

*If \( p_t \) is bounded for some \( t > 0 \) then*

\[
\int_{0^+}^\infty \frac{p_t(0)}{t} dt < \infty.
\]

*If \( p_t \) is bounded for every \( t > 0 \) then*

\[
\int_{0^+}^\infty \frac{p_t(0)}{t} dt = \infty.
\]

Thus, by Fatou Lemma and \( h'(x) = \int_0^\infty q^*_t(x)dt \), we get

\[
\lim \inf_{x \to 0^+} \frac{h'(x)}{h(x)} \geq \int_0^\infty \lim \inf_{x \to 0^+} \frac{q^*_t(x)}{h(x)} dt = \int_0^\infty \frac{p_t(0)}{t} dt = \infty.
\]
Step III: upper-bounds for $q^*_t(x)/h(x)$

Proposition (L. Chaumont, JM 2015)

There exists $c > 0$ such that

$$\frac{q^*_t(x)}{h(x)} \leq c \frac{p_{t/6}^S(0)}{t}, \quad x > 0, t > 0.$$ 

Here

$$p_t^S = p_t * \tilde{p}_t$$

is the transition density function of the symmetrization of $X$.

Obviously, under $(H1)$, $p_t^S$ is also bounded for every $t > 0$.

Consequently, the above-given upper-bounds are integrable at infinity (in $t$).
Step III: upper-bounds for $q_t^*(x)/h(x)$

Since $e^{-t\psi(\xi)}$ is integrable

$$q_t^*(x, y) \leq p_t(y - x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(x-y)} e^{-t\psi(\xi)} d\xi \leq p_{t/2}^S(0).$$

Moreover (M. Kwasnicki, JM, M. Ryznar, AoP 2013)

$$\int_0^\infty q_t(x, y)dy = \mathbb{P}(X_t < x) \leq \frac{e}{e - 1} \kappa(1/t, 0)h(x)$$

$$\int_0^\infty q_t^*(x, y)dy \leq \frac{e}{e - 1} \kappa^*(1/t, 0)h^*(x).$$
Thus, by Chapman-Kolmogorov equation

\[ q^*_3(t, x, y) = \int_0^\infty \int_0^\infty q^*_t(x, z)q^*_t(z, w)q^*_t(w, y) \, dz \, dw \]

\[ \leq p^S_{t/2}(0) \int_0^\infty q^*_t(x, z) \, dz \int_0^\infty q^*_t(w, y) \, dw \]

\[ = p^S_{t/2}(0) \int_0^\infty q^*_t(x, z) \, dz \int_0^\infty q_t(y, w) \, dw \]

\[ \leq \left( \frac{e}{e - 1} \right)^2 p^S_{t/2}(0) h^*(x)h(y)\kappa(1/t, 0)\kappa^*(1/t, 0) \]

\[ = \left( \frac{e}{e - 1} \right)^2 \frac{p^S_{t/2}(0)}{t} h^*(x)h(y). \]

Consequently, for every \( y > 0 \) and \( t > 0 \) we have

\[ \frac{q^*_t(y)}{h(y)} = \lim_{x \to 0^+} \frac{q^*_t(x, y)}{h^*(x)h(y)} \leq 3 \left( \frac{e}{e - 1} \right)^2 \frac{p^S_{t/6}(0)}{t}. \]
Writing once again

\[
\frac{f_t(x)}{h'(x)} = \int_0^\delta n(t - s < \zeta) \frac{q_s^*(x)}{h'(x)} ds + \frac{h(x)}{h'(x)} \int_\delta^t n(t - s < \zeta) \frac{q_s^*(x)}{h(x)} ds \\
+ \frac{h(x)}{h'(x)} \frac{q_t^*(x)}{h(x)}
\]

we can see that

\[
\limsup_{x \to 0^+} \frac{f_t(x)}{h'(x)} \leq (n(t < \zeta) - \varepsilon) \limsup_{x \to 0^+} \frac{1}{h'(x)} \int_0^\delta q_t^*(x) dx \\
\leq n(t < \zeta) - \varepsilon
\]
To deal with lower bounds, we use monotonicity of \( n(t < \zeta) \) and we write

\[
f_t(x) = \int_0^t q^*_s(x)n(t - s < \zeta)ds + dq^*_t(x)
\]

\[
\geq n(t < \zeta) \int_0^t q^*_s(x)ds
\]

and consequently

\[
\liminf_{x \to 0^+} \frac{f_t(x)}{h'(x)} \geq n(t < \zeta) - \limsup_{x \to \infty} \frac{h(x)}{h'(x)} \int_t^\infty \frac{q^*_s(x)}{h'(x)} ds
\]

\[
\geq n(t < \zeta)
\]
Thank you very much.