INFINITESIMAL GENERATORS OF POLYNOMIAL PROCESSES

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Plan

1. Algebra for polynomial processes
2. Quadratic harnesses
3. QH flow
4. Free QH: $\gamma = -\tau\sigma$

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Algebra of sequences of polynomials

Let $\mathcal{Q}$ be a linear space of sequences of polynomials in variable $x \in \mathbb{R}$. For $\mathbb{P} = (p_0, p_1, \ldots,)$ and $\mathbb{Q} = (q_0, q_1, \ldots)$ we define $\mathbb{P}\,\mathbb{Q} =: \mathbb{R} = (r_0, r_1, \ldots) \in \mathcal{Q}$ where

$$r_k = \sum_j [q_k]_j \, p_j, \qquad k = 0, 1, \ldots$$

This product is associative. The identity is

$$\mathbb{E} = (1, x, x^2, \ldots).$$

If $\deg(p_n) = n$ for all $n \geq 0$ then $\mathbb{P} = (p_0, p_1, p_2, \ldots)$ is invertible.

$\mathbb{F}$ and $\mathbb{D}$

We will need special elements $\mathbb{F}$, $\mathbb{D} \in \mathcal{Q}$:

$$\mathbb{F} = (x, x^2, x^3, x^4 \dots) \qquad \text{and} \qquad \mathbb{D} = (0, 1, x, x^2, \dots).$$

We note that

$$\mathbb{D}\mathbb{F} = \mathbb{E} \qquad \text{and} \qquad \mathbb{E} - \mathbb{F}\mathbb{D} = (1, 0, 0, 0, \dots)$$

Isomorphism

Let $\mathcal{P}$ denote the space of all polynomials on $\mathbb{R}$.

The algebra $\mathrm{End}(\mathcal{P})$ is isomorphic to the algebra $\mathcal{Q}$. The isomorphism $\Psi : \mathrm{End}(\mathcal{P}) \to \mathcal{Q}$ is defined by

$$\Psi(\mathbf{P}) = (\mathbf{P}(1), \mathbf{P}(x), \mathbf{P}(x^2), \ldots) := \mathbb{P} \in \mathcal{Q}, \qquad \forall\, \mathbf{P} \in \mathrm{End}(\mathcal{P})$$

For $\mathbf{P}$, $\mathbf{Q} \in \mathrm{End}(\mathcal{P})$ and $\mathbb{P} = \Psi(\mathbf{P})$, $\mathbb{Q} = \Psi(\mathbf{Q})$

$$\Psi(\mathbf{P} \circ \mathbf{Q}) = \mathbb{P}\mathbb{Q}.$$

Polynomial process

Let $\mathcal{P}_{\leq k}$ be a space of polynomials on $\mathbb{R}$ with degree at most k, $k = 0, 1, 2, \ldots$

Let $X = (X_t)_{t \geq 0}$ be a non-homogeneous Markov process with infinite state space $S \subset \mathbb{R}$. If

$$\mathrm{E}(f(X_t)|X_s) \in \mathcal{P}_{\leq k}$$

for any $f \in \mathcal{P}_{\leq k}$, $k = 1, 2, \ldots$, we call X a *polynomial process*.

Family of operators on $\mathcal{P}$

Through

$$\mathbf{P}_{s,t} f(x) := \mathrm{E}(f(X_t)|X_s = x), \qquad f \in \mathcal{P},$$

such process can be identified with a family of linear operators $\mathbf{P}_{s,t} : \mathcal{P} \to \mathcal{P}$, $0 \le s \le t$, satisfying

- $\mathbf{P}_{s,t}(\mathcal{P}_{\le k}) = \mathcal{P}_{\le k}$, $k \ge 0$;
- $\mathbf{P}_{s,t}(1) = 1$;
- for $0 \le s \le t \le u$

$$\mathbf{P}_{s,t} \circ \mathbf{P}_{t,u} = \mathbf{P}_{s,u}.$$

Family of elements of $\mathcal{Q}$

Through isomorphism $\Psi : \mathrm{End}(\mathcal{P}) \to \mathcal{Q}$ the process X can be identified with a family of $\mathbb{P}_{s,t} = (p^0_{s,t}, p^1_{s,t}, p^2_{s,t}, \dots) \in \mathcal{Q}$, $0 \leq s \leq t$ satisfying

- $\mathbb{P}_{s,t}$ is invertible;
- $\mathbb{P}_{s,t}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}$;
- for $0 \leq s \leq t \leq u$

$$\mathbb{P}_{s,t}\mathbb{P}_{t,u} = \mathbb{P}_{s,u}. \tag{1}$$

Martingale polynomials

From the above conditions we conclude that

- $\mathbb{P}_{t,t} = \mathbb{E}$
- if $\mathbb{M}_t = \mathbb{P}_{0,t}^{-1}$ then

$$\mathbb{M}_s = \mathbb{P}_{s,t}\mathbb{M}_t. \tag{2}$$

The identity (2) follows by multiplying the flow equation

$$\mathbb{P}_{0,s}\mathbb{P}_{s,t} = \mathbb{P}_{0,t}$$

by $\mathbb{M}_s$ and $\mathbb{M}_t$:

$$\mathbb{M}_s\mathbb{P}_{0,s}\mathbb{P}_{s,t}\mathbb{M}_t = \mathbb{M}_s\mathbb{P}_{0,t}\mathbb{M}_t.$$

Martingale polynomials, cont.

Condition (2) says that if $\mathbb{M}_t = (m_t^0,\ m_t^1,\ m_t^2, \ldots)$ then $(m_t^k)_{t\geq 0}$, $k \geq 0$, are martingale polynomials for X, that is

$$\mathrm{E}(m_t^k(X_t)|X_s) = m_s^k(X_s), \qquad 0 \leq s \leq t \quad k \geq 0.$$

Note that $\deg(m_t^k) = k$, $k \geq 0$.

Note also that

$$\mathbb{P}_{s,t} = \mathbb{M}_s \mathbb{M}_t^{-1}$$

Harness (Hammersley, 1967, Mansuy, Yor, 2005)

Let $X = (X_t)_{t \geq 0}$ be a real valued stochastic process,

$$\mathrm{E}\, X_t = 0, \qquad \mathrm{E}\, X_s X_t = s \qquad \forall\, 0 \leq s \leq t.$$

Let $(\mathcal{F}_{s,u})_{0 \leq s < u}$ be a natural **past-future filtration** of X, i.e.

$$\mathcal{F}_{s,u} = \sigma\{X_\alpha,\ \alpha \notin (s,u)\}.$$

The process X is a **harness** if $\forall\, 0 \leq s < t < u$

$$\mathrm{E}(X_t | \mathcal{F}_{s,u}) = a_{tsu} X_s + b_{tsu} X_u = \frac{(u-t)X_s + (t-s)X_u}{u-s} = t\Delta_{s,u} + \widetilde{\Delta}_{s,u},$$

where

$$\Delta_{s,u} = \frac{X_u - X_s}{u-s} \qquad \text{oraz} \qquad \widetilde{\Delta}_{s,u} = \frac{\frac{1}{u}X_u - \frac{1}{s}X_s}{\frac{1}{u} - \frac{1}{s}}.$$

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Quadratic harness (BMW, 2007)

If X is a harness and additionally

$$\mathrm{E}(X_t^2|\mathcal{F}_{s,u}) = A_{tsu}X_s^2 + B_{tsu}X_sX_u + C_{tsu}X_u^2 + D_{tsu}X_s + E_{tsu}X_u + F_{tsu}$$

then X is called a **quadratic harness**.

Then there exist numbers $\theta,\ \eta \in \mathbb{R},\ \tau,\ \sigma \geq 0,\ \gamma \leq 1 + 2\sqrt{\tau\sigma}$ such that

$$\mathrm{Var}(X_t|\mathcal{F}_{s,u}) = F_{tsu}\,K\left(\Delta_{s,u},\,\widetilde{\Delta}_{s,u}\right),$$

where

$$K(x,y) := 1 + \theta x + \eta y + \tau x^2 + \sigma y^2 - (1-\gamma)xy$$

and $\quad F_{tsu} = \frac{(u-t)(u-s)}{u(1+\sigma s)+\tau-\gamma s}.$

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Typical properties

- linear conditional means and quadratic conditional variances (as above, except the case $\gamma = 1 + 2\sqrt{\tau\sigma}$) uniquely determine all moments, and these moments uniquely determine the process $X \sim \mathrm{QH}(\theta, \eta, \tau, \sigma; \gamma)$;
- X is a (non-homogeneous) Markov process;
- X has orthogonal martingale polynomials $(m_n(\cdot, t))$ i.e. for $t \geq 0$

$$\mathrm{E}\, m_n(X_t, t)\, m_k(X_t, t) = 0, \quad k \neq n$$

and for $n \geq 0$

$$\mathrm{E}(m_n(X_t, t) | \mathcal{F}_s) = m_n(X_s, s) \quad s < t.$$

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Examples of $X \sim \mathrm{QH}(\theta, \eta, \tau, \sigma, \gamma)$

- $\mathrm{QH}(0,\ 0,\ 0,\ 0,\ 1)$ - Wiener process,
- $\mathrm{QH}(\theta,\ 0,\ 0,\ 0,\ 1)$ - centered Poisson process,
- $\mathrm{QH}(\theta,\ 0,\ \tau,\ 0,\ 1)$ - Lévy-Meixner processes (Schoutens, 2000),
- $\mathrm{QH}(0,\ 0,\ 0,\ 0,\ 0)$ - free Brownian motion (Biane, 1998),
- $\mathrm{QH}(0,\ 0,\ 0,\ 0,\ q)$ - q-Gaussian process (Bożejko, Kümmerer, Speicher, 1997),
- $\mathrm{QH}(\theta,\ 0,\ \tau,\ 0,\ 0)$ - free Lévy-Meixner process (Anshelevich, 2003),
- $\mathrm{QH}(\theta,\ 0,\ \tau,\ 0,\ q)$ - q-Lévy-Meixner processes (BW, 2005)
- $\mathrm{QH}(\theta,\ \eta,\ 0,\ 0,\ q)$ - bi-Poisson processes (Biane, 1996, BW, 2006 ($q = 1$), BW 2007 ($q = 0$), BMW, 2008 ($q \in [-1, 1]$),
- $\mathrm{QH}(\theta,\ \eta,\ \tau,\ \sigma,\ -\tau\sigma)$ - free quadratic harness (BMW, 2011).

Definition

$(\mathbb{P}_{s,t},\ 0 \leq s \leq t)$ is a quadratic harness $QH(\theta, \eta, \tau, \sigma\gamma)$ flow if

1. (martingale) $\mathbb{P}_{s,t}(\mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2) = \mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2$,
2. (harness) There exists $\mathbb{X} \in \mathcal{Q}$ such that

$$\mathbb{P}_{0,t}\mathbb{F} = (\mathbb{F} + t\mathbb{X})\mathbb{P}_{0,t},$$

3. (quadratic harness) The above $\mathbb{X}$ satisfies

$$\mathbb{X}\mathbb{F} - \gamma\mathbb{F}\mathbb{X} = \mathbb{E} + \theta\mathbb{X} + \eta\mathbb{F} + \tau\mathbb{X}^2 + \sigma\mathbb{F}^2.$$

Martingale and harness

Ad.1. Note that $\mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2 = (0, x, 0, 0, \ldots)$. That is $\mathbf{P}_{s,t}(x) = x$, meaning that $\mathrm{E}(X_t|\mathcal{F}_s) = X_s$.

Ad.2. For a martingale polynomial $m_n(\cdot, t)$ the harness property gives

$$\mathrm{E}(X_t m_n(X_u, u)|X_s) = \tfrac{u-t}{u-s} X_s m_n(X_s, s) + \tfrac{t-s}{u-s}\mathrm{E}(X_u m_n(X_u, u)|X_s).$$

Martingality gives

$$\mathrm{E}(X_t m_n(X_u, u)|X_s) = \mathrm{E}(X_t m_n(X_t, t)|X_s).$$

That is

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Ad 2, cont.

Equivalently,

$$\mathbb{P}_{s,t}\mathbb{F}\mathbb{M}_t = \frac{u-t}{u-s}\mathbb{F}\mathbb{M}_s + \frac{t-s}{u-s}\mathbb{P}_{s,u}\mathbb{F}\mathbb{M}_u. \tag{3}$$

Define $\mathbb{J}_t$ by $\mathbb{F}\mathbb{M}_t = \mathbb{M}_t\mathbb{J}_t$. Then from (3) we get

$$(u-s)\mathbb{J}_t = (u-t)\mathbb{J}_s + (t-s)\mathbb{J}_u$$

and thus

$$\mathbb{J}_t = \mathbb{Y} + t\mathbb{X} \qquad \text{for some} \quad \mathbb{X}, \mathbb{Y} \in \mathcal{Q}.$$

Consequently,

$$\mathbb{F}\mathbb{M}_t = \mathbb{M}_t(\mathbb{Y} + t\mathbb{X}).$$

Since $\mathbb{M}_0 = \mathbb{E}$ we get that $\mathbb{Y} = \mathbb{F}$.

Generator

The left infinitesimal generator $\mathbb{A}_t^-$ exists since $\mathbb{M}_t$ is element-wise differentiable in t

$$\mathbb{A}_t^- \,\mathbb{M}_t = \lim_{h \to 0^+} \frac{1}{h}\left(\mathbb{P}_{t-h,t} - \mathbb{E}\right)\mathbb{M}_t = \lim_{h \to 0^+} \frac{1}{h}(\mathbb{M}_{t-h} - \mathbb{M}_t) = -\frac{\partial}{\partial t}\,\mathbb{M}_t.$$

The right infinitesimal generator exists since $\mathbb{P}_{s,t}$ is continuous in t

$$\mathbb{A}_t^+ \mathbb{M}_t = \lim_{h \to 0^+} \frac{1}{h}(\mathbb{P}_{t,t+h}\mathbb{M}_t - \mathbb{M}_t) = \lim_{h \to 0^+} \mathbb{P}_{t,t+h}(\mathbb{M}_t - \mathbb{M}_{t+h})$$

$$= \lim_{h \to 0^+} \mathbb{P}_{t,t+h} \lim_{h \to 0^+} \frac{1}{h}\left(\mathbb{M}_t - \mathbb{M}_{t+h}\right) = -\frac{\partial}{\partial t}\,\mathbb{M}_t$$

Consequently,

$$\mathbb{A}_t^- = \mathbb{A}_t^+ := \mathbb{A}_t \in \mathcal{Q}.$$

Uniqueness

We note that for $\mathbb{A}_t = (a_t^0, a_t^1, a_t^2, \dots)$

- $a_t^n \in \mathcal{P}_n$
- $a_t^0 = 0$ since $\mathbb{A}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = 0$ (due to $\mathbb{P}_{s,t}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}$);
- $a_t^1 = 0$ since $\mathbb{A}_t(\mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2) = 0$ (due to $\mathbb{P}_{s,t}(\mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2) = \mathbb{F}\mathbb{D} - \mathbb{F}^2\mathbb{D}^2$).

Proposition. *A polynomial process $\{\mathbb{P}_{s,t},\ 0 \le s \le t\}$ satisfying the harness property*

$$\mathbb{P}_{0,t}\mathbb{F} = (\mathbb{F} + t\mathbb{X})\mathbb{P}_{0,t}$$

is uniquely determined by the generators $(\mathbb{A}_t)_{t\ge 0}$.

Auxiliary sequence $\mathbb{H}_t \in \mathcal{Q}$

For a polynomial harness $\{\mathbb{P}_{s,t}\}$, differentiate wrt t

$$\mathbb{F}\mathbb{M}_t = \mathbb{M}_t(\mathbb{F} + t\mathbb{X}), \qquad \mathbb{X} \in \mathcal{Q},$$

to see that

$$\mathbb{A}_t\mathbb{F} - \mathbb{F}\mathbb{A}_t = \mathbb{M}_t\mathbb{X}\mathbb{M}_t^{-1} =: \mathbb{H}_t.$$

Proposition. $\mathbb{H}_t$ *uniquely determines* $\mathbb{A}_t$.

Proof. Since $\mathbb{A}_t = \mathbb{A}_t\mathbb{F}\mathbb{D}$ we have

$$\mathbb{A}_t = \mathbb{F}\mathbb{A}_t\mathbb{D} + \mathbb{H}_t\mathbb{D}.$$

Iterating this we get

$$\mathbb{A}_t = \sum_{k=0}^{\infty} \mathbb{F}^k\mathbb{H}_t\mathbb{D}^{k+1}$$

well defined as element-wise it is a sum of finitely many elements.

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QH process

Theorem. *Let* $\{\mathbb{P}_{s,t},\ 0 \le s \le t\}$ *be a quadratic harness. Denote* $\mathbb{T}_t = \mathbb{F} - t\mathbb{H}_t$. *Then*

$$\mathbb{H}_t\mathbb{T}_t - \gamma\mathbb{T}_t\mathbb{H}_t = \mathbb{E} + \theta\mathbb{H}_t + \eta\mathbb{T}_t + \tau\mathbb{H}_t^2 + \sigma\mathbb{T}_t^2. \tag{4}$$

It follows by the QH property together with

$$\mathbb{H}_t = \mathbb{M}_t\mathbb{X}\mathbb{M}_t^{-1} \qquad \text{and} \qquad \mathbb{T}_t = \mathbb{M}_t\mathbb{F}\mathbb{M}_t^{-1}.$$

If

$$\gamma \le 1, \quad \tau, \sigma \le 0, \quad \text{and} \quad \tau\sigma \ne 0$$

the equation (4) has a unique solution $\mathbb{H}_t \in \mathcal{Q}$ such that $h_0 = 0$.

Main goal: Generator $\mathbf{A}_t$ of free QH

The commutation equation has the form

$$(1+\sigma t)\mathbb{H}_t\mathbb{F} = \mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2+(\theta-\eta t)\mathbb{H}_t-\sigma(t+\tau)\mathbb{F}\mathbb{H}_t+(t+\tau)(1+\sigma t)\mathbb{H}_t^2.$$

Its solution is

$$\mathbb{H}_t = \frac{1}{1+\sigma t}\left(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2\right)\phi_t(\mathbb{D})\mathbb{D},$$

where $\phi_t(z) = \sum_{k=1}^{\infty} c_k(t) z^{k-1}$ is a power series defined (at least in a neighbourhood of zero) by

$$(z^2+\eta z+\sigma)(t+\tau)\phi_t^2 + ((\theta-\eta t)z - 2t\sigma - \sigma\tau - 1)\phi_t + t\sigma + 1 = 0$$

and $\phi_t(0) = 1$.

Back to the generator

$$\mathbb{A}_t = \sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{H}_t \mathbb{D}^{k+1} = \frac{1}{1+t\sigma} \sum_{k=0}^{\infty} \mathbb{F}^k (\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\phi_t(\mathbb{D})\mathbb{D}\mathbb{D}^{k+1}$$

$$= \frac{1}{1+t\sigma} (\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2) \left(\sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{D}^{k+1} \right) \phi_t(\mathbb{D})\mathbb{D}.$$

But

$$\sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{D}^{k+1} = (0, 1, 2x, 3x^2, \ldots, kx^{k-1}, \ldots) =: \mathbb{D}_1.$$

That is

$$\mathbb{A}_t = \frac{1}{1+t\sigma} (\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\, \mathbb{D}_1\, \phi_t(\mathbb{D})\, \mathbb{D}.$$

Who is ϕ_t?

Let $G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\,\mu(dx)$, $z \in \mathbb{C}_+$, denotes the Cauchy-Stieltjes transform of a measure μ. It is well known that G_μ is an analytic function in $\mathbb{C}_+$, determines μ uniquely. If it extends to real z with $|z|$ large enough then the corresponding moment generating function M_μ is well defined for $|z|$ small enough and

$$G_\mu(z) = \frac{1}{z} M_\mu(\tfrac{1}{z}).$$

It appears that $\phi_t(1/z)/z$ agrees with the Cauchy-Stieltjes transform of a probability measure ν_t identified in Saitoh and Yosida (2001).

Operator $\mathbf{H}_t$ through the isomorphism Ψ

For $\mathbf{H}_t$ such that $\Psi(\mathbf{H}_t) = \mathbb{H}_t$ we have

$$\mathbf{H}_t(x^n) = \frac{1+\eta x+\sigma x^2}{1+\sigma t} \sum_{k=1}^{n} c_k(t)\, x^{n-k}.$$

Since $c_k(t) = \int y^{k-1}\, \nu_t(dy)$ we get

$$\mathbf{H}_t(f)(x) = \frac{1+\eta x+\sigma x^2}{1+\sigma t} \int \frac{f(x)-f(y)}{y-x}\, \nu_t(dy), \qquad f \in \mathcal{P}.$$

Operator $\mathbf{A}_t$

Since

$$\mathbf{A}_t(x^{n+1}) = \mathbf{H}_t(x^n) + x\mathbf{A}_t(x^n)$$

we get

$$\mathbf{A}_t(f)(x) = \frac{1+\eta x+\sigma x^2}{1+\sigma t} \int \frac{\partial}{\partial x} \left(\frac{f(x)-f(y)}{y-x} \right) \nu_t(dy),$$

$f \in \mathcal{P}$.

Who is ν_t?

If $\mu_2(dx) = (ax^2 + bx + c)\mu_1(dx)$ and $\int x\mu_1(dx) = m$, then

$$G_{\mu_2}(z) = (az^2 + bz + c)G_{\mu_1}(z) - a(m + z) - b. \qquad (5)$$

Let $\mu_1 = \pi_{t,\eta,\theta,\sigma,\tau}$ be the univariate law of X_t. Then (5) with

$$m = 0, \qquad (a, b, c) = \frac{1}{t(t+\tau)}(\tau, \theta t, t^2)$$

gives

$$\phi_t(z) = G_{\mu_2}(1/z)/z.$$

Therefore, ϕ_t is a moment generating function of the probability measure

$$\nu_t(dx) = \frac{t^2 + \theta tx + \tau x^2}{t(t+\tau)} \pi_{t,\eta,\theta,\sigma,\tau}(dx).$$