Non-local Hamilton–Jacobi–Bellman equations on Hilbert spaces

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Controlled system

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$X(t) = x \in H,\ s \in [t, T],\ Z$ Lévy process on U, ν-Lévy measure
$a(s), \quad s \in [t, T],$ control, $a(s) \in \Lambda$-set of control parameters

$$V(t,x) = \inf_{a(\cdot)} \mathbb{E}\left(\int_t^T f(X(s), a(s))\, ds + h(X(T)) \right), \tag{2}$$

$t \in [0, T]$, $\quad x \in H$, $\quad$ value function.

HJB equation:

$$\frac{\partial u}{\partial t}(t,x) + \inf_{a\in\Lambda}(L^a u(t,x) + f(x,a)) = 0,$$
$$u(T,x) = h(x), \ \ x \in H,$$

(3)

$$L^a v(x) = \langle Ax, Dv(x)\rangle + \langle g(x,a), Dv(x)\rangle$$
$$+ \int_U [v(x + G(x,a)z) - v(x) - \langle Dv(x), G(x,a)z\rangle]\, \nu\, dz(s)$$

(4)

$$L^a v(x) = \langle Ax, Dv(x)\rangle + L_0^a v(x),$$

$$(5) \quad \frac{\partial u}{\partial t}(t,x) + \langle Ax, Du(t,x)\rangle + \inf_{a\in\Lambda}(L_0^a u(t,x) + f(x,a)) = 0,$$
$$u(T,x) = h(x).$$

Meta theorem:

V is the unique solution to (3) and $\hat{a}(t,x)$, $t \in [0,T]$, $x \in H$

$$\inf_{a \in \Lambda}(L^a V(t,x) + f(x,a)) = L^{\hat{a}(t,x)} V(t,x) + f(x, \hat{a}(t,x))$$

is the optimal feedback strategy. Process $\widehat{X}$:

$$
\begin{aligned}
d\widehat{X}(s) &= \left[A\widehat{X}(s) + g(\widehat{X}(s), \hat{a}(s, \widehat{X}(s)))\right] ds \\
&\qquad + G(\widehat{X}(s-), \hat{a}(s, \widehat{X}(s-))) \, dZ(s), \\
\widehat{X}(t) &= x
\end{aligned}
$$

is an optimal one.

α-Stable systems

$$dX(s) = (AX(s) + g(X(s), a(s)))ds + dZ(s), \tag{6}$$

$$Ae_n = -\lambda_n e_n, \quad n = 1, 2, \ldots \qquad Z(s) = \sum_{n=1}^{+\infty} \beta_n Z_n(s) e_n,$$

Z_n independent, α-stable processes

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) = \langle Ax, Dv(t,x)\rangle + \int_H (v(t,x+z) \\ - v(t,x) - \langle Dv(t,x), z\rangle)\nu(dz) \\ + \inf_{a\in\Lambda}\langle g(x,a), Dv(t,x)\rangle.\end{aligned}$$

Mild HJB equation

$$\frac{\partial v}{\partial t}(t,x) = \mathcal{A}v(t,x) + \mathcal{H}(x, Dv(t,x)) \tag{7}$$
$$v(0,x) = h(x)$$

$\mathcal{H}(x,p) = \inf_{a\in\Lambda}\langle g(x,a), p\rangle$, $\mathcal{A}$ generator of

$$dY(s) = AY(s)\,ds + dZ(s),$$

P_t transition semigroup of Y,

$$v(t,x) = P_t h(x) + \int_0^t P_{t-s}[\mathcal{H}(\cdot), Dv(s,\cdot)]\,ds. \tag{8}$$

Existence theorem

Theorem 1

Assume $\alpha \in (1,2)$. Under $(\mathrm{A1})$, $(\mathrm{A2})$ equation (8) has unique solution in $C_0^{1,\gamma}([0,T],H)$

$$(\mathrm{A1}) \quad \sum_{n=1}^{+\infty} \frac{\beta_n^\alpha}{\lambda_n} < +\infty, \qquad \frac{\beta_n^\alpha}{\lambda_n} \geq c\frac{1}{\lambda_n^{\alpha\gamma}}, \ n = 1,\ldots,\gamma \in (0,1),$$

$(\mathrm{A2})$ If $\psi \in C_b(H,H)$ then $\mathcal{H}(\cdot,\psi(\cdot)) \in C_b(H,H)$.

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$$\|u\|_{C_0^{1,\gamma}} = \sup_{0\leq t\leq T}\left[\sup_x |u(t,x)| + t^\gamma \sup_x \|Du(t,x)\|\right]$$

Test functions

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \inf_{a\in\Lambda}\left[L^a u(t,x) + f(x,a)\right] = 0, \\ &u(T,x) = h(x), \;\; t\in[0,T], \;\; x\in H. \end{aligned} \tag{9}$$

$\varphi(t,x)$, $t\in[0,T]$, $x\in H$ is a test function if φ is continuous on $(0,T)\times H$ and
φ_t, $D\varphi$, $A^* D\varphi$ $D^2\varphi$ are uniformly continuous on $(\varepsilon, T-\varepsilon)\times H$ for every $\varepsilon > 0$ and locally bounded.

Subsolution

A continuous function $u : (0, T] \times H \to R$ is a viscosity subsolution to (9) if $u(T, x) \leq h(x)$ and whenever $u - \varphi$ has a global maximum at a point (t, x) for a test function φ then

$$\frac{\partial \varphi}{\partial t}(t, x) + \inf_{a \in \Lambda} \left[L^a \varphi(t, x) + f(x, a) \right] \geq 0. \tag{10}$$

Supersolution and viscosity solution

A continuous function $u : (0, T] \times H \to R$ is a viscosity supersolution to (9) if $u(T, x) \geq h(x)$ and whenever $u - \varphi$ has a global minimum at a point (t, x) for a test function φ then

$$\frac{\partial \varphi}{\partial t}(t, x) + \inf_{a \in \Lambda} \left[L^a \varphi(t, x) + f(x, a) \right] \leq 0. \tag{11}$$

A viscosity solution to (9) is a function which is both subsolution and supersolution.

Assumptions

(A3) $\|g(x,a) - g(y,a)\| + \|G(x,a) - G(y,a)\| \leq C\|x-y\|$,

(A4) $\|g(0,a)\| + \|G(0,a)\| \leq C$,

(A5) $|f(x,a) - f(y,a)\| + \|h(x) - h(y)\| \leq \sigma(\|x-y\|)$,
f, h continuous, σ modulus of confinity

(A6) $\int_U \|z\|^2 \nu(dz) < +\infty$.

Existence theorem

Theorem 2

Under $(\mathrm{A3})$–$(\mathrm{A6})$ the value function V is a viscosity solution to (9).

B-continuity

Let B-positive definite, bounded operator on H such that for some c_0:

$$\langle(-A^*B + c_0B)x, x\rangle \geq 0, \qquad \text{for all } x \in B.$$

Define

$$\|x\|_B = \|B^{1/2}x\| = \langle Bx, x\rangle^{1/2},$$

u is B-continuous on $(0, T] \times H$ if whenever $t_n \to t$, $Bx_n \to Bx$ and x_n bounded, $u(t_n, x_n) \to u(t, x)$.

Uniqueness

$$(\mathrm{A3})' \qquad \|g(x,a)-g(y,a)\| + \|G(x,a)-G(y,a)\| \le C\|x-y\|_B$$

$$(\mathrm{A5})' \qquad |f(x,a)-f(y,a)| + \|h(x)-h(y)| \le \sigma(\|x-y\|_B)$$

B-test functions, B-viscosity solution is B-continuous.

Theorem 3

Under $(\mathrm{A3})'$, $(\mathrm{A4})$, $(\mathrm{A5})'$, $(\mathrm{A6})$ equation (9) has unique B-viscosity solution. It is identical with the value function.

Wave equation

$$\frac{\partial^2 x}{\partial s^2} = \left(\Delta x(s,\xi) + \bar{g}(x(s,\xi), a(s))\right) ds + \overline{G}(x(s-,\xi), a(s))\, dZ(s).$$

Initial and boundary conditions, $s \in [t, T]$:

$$x(t,\xi) = \bar{x}(\xi), \qquad \frac{\partial x}{\partial t}(s,\xi) = y(s,\xi) = 0, \qquad \xi \in \partial\mathcal{O},$$

$x(s,\xi)$ – position, $\frac{\partial x}{\partial s}(s,\xi) = y(s,\xi)$ – velocity

$$X(s) = \begin{pmatrix} x(s,\cdot) \\ y(s,\cdot) \end{pmatrix} \in H = \begin{pmatrix} H_0^1(\mathcal{O}) \\ \times \\ L^2(\mathcal{O}) \end{pmatrix}$$

$$A = \begin{pmatrix} 0, & I \\ \Delta, & 0 \end{pmatrix}, \qquad D(A) = \begin{pmatrix} H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \\ \times \\ H_0^1(\mathcal{O}) \end{pmatrix}$$

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}, a\right) = \begin{pmatrix} 0 \\ \bar{g}(x, a) \end{pmatrix}, \qquad G\left(\begin{pmatrix} x \\ y \end{pmatrix}, a\right) = \begin{pmatrix} 0 \\ \overline{G}(y, a) \end{pmatrix}$$

$$B = \begin{pmatrix} (-\Delta)^{-\frac{1}{2}}, & 0 \\ 0, & (-\Delta)^{-\frac{1}{2}} \end{pmatrix}$$

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_B = \left(\|(-\Delta)^{-\frac{1}{4}} x\|^2_{L^2(\mathcal{O})} + \|(-\Delta)^{-\frac{1}{4}} y\|^2_{L^2(\mathcal{O})} \right)^{1/2}$$

Z – one dimensional Lévy process.

Gradient estimates:

$$dY(s) = AY(s)\,ds + dZ(s), \qquad Y(0) = x, \;\; Y(s,x)$$

$$P_t\varphi(x) = \mathbb{E}(\varphi(Y(s,x))),$$

$$\sup_{x\in H} \|DP_t\varphi\| \le c_\alpha C_t \sup_{x\in H} |\varphi(x)|, \qquad t > 0,$$

$$c_\alpha = \int \frac{(P'_\alpha(z))^2}{P_\alpha(z)}\,dz, \qquad C_t = \sup_n \frac{e^{-\lambda_n t}\lambda n^{1/2}}{\beta n}$$

Enrico Priola, J. Z. (PTRF, 149 (2011), 97–137).

Dynamic Programming Principle

Yosida approximations

$$V(t,x) = \inf_{(a(\cdot),\tau_{a(\cdot)})} \mathbb{E}\left[\int_t^{\tau_{a(\cdot)}} f(X(s),a(s))\,ds + V(\tau_{a(\cdot)}, X(\tau_{a(\cdot)}))\right]$$

$$t \le \tau_{a(\cdot)} \le T, \quad \text{stopping time}$$

Dubling technique of Crandall and Lions

Show that if u, v are subsolution and supersolution then

$$u(t,x) - v(t,x) \leq 0, \qquad t \in [0,T],\ x \in H,$$

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$$u(t,x) - v(t,x) \leq 0, \qquad t \in [0,T],\ x \in H,$$

$$\phi_\gamma((t,x),(s,y)) = u(t,x) - v(s,y) - \psi_\gamma(t,z,s,y), \quad \gamma\text{-parameters}$$

$$\gamma = (\varepsilon,\beta), \qquad \psi_\gamma(t,x,s,y) = \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \frac{(t-s)^2}{2\beta},$$

$(\bar{t},\bar{x},\bar{s},\bar{y}) \in (0,T] \times H \times (0,T] \times H$ maximizer of ϕ_γ. Then

$$u(t,x) - v(t,x) = \phi_\gamma((t,x),(t,x)) \leq \phi_\gamma((\bar{t},\bar{x}),(\bar{s},\bar{y})).$$

Show that $\phi_\gamma((\bar{t},\bar{x}),(\bar{s},\bar{y}))$ is small for proper choice of γ.

Pushing maximas

Consider map

$$(t, x) \to \phi_\gamma((t, x), (\bar{s}, \bar{y}))$$

which attains maximum at $(\bar{t}, \bar{x})$; subsolution property of u gives relations between $(\bar{t}, \bar{x}), (\bar{s}, \bar{y})$.
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Comparison property and maximum principle

$$\frac{d}{dt}u(t,x) = \mathcal{A}(u(t,x)), \quad u(0,x) = x, \quad x \in C(E). \tag{12}$$

Equation (12) has comparison property iff $x \leq y$ implies $u(t,x) \leq u(t,y), \ t \geq 0$.

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Operator $\mathcal{A}$ has maximum property iff $x \leq y$ and $x(\xi) = y(\xi)$ imply $\mathcal{A}x(\xi) \leq \mathcal{A}y(\xi), \quad \xi \in E$.

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Fact: Comparison property and maximum principle are equivalent.

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