Analytical, geometric, and stochastic properties of a class of infinite dimensional stochastic processes with unbounded diffusion

John Karlsson and Jörg-Uwe Löbus

University of Linköping

Probability and Analysis, Bedlewo, May 4-8, 2015
1 Motivation and general theory
2 Closability
3 Quasi-regularity
4 Moving to a geometric setting
5 Ornstein-Uhlenbeck processes with unbounded diffusion
6 References
Unbounded diffusion – an example

A formal SPDE

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, be an in general unbounded real sequence. Let $S_i, i \in \mathbb{N}$, denote the ONB in the Cameron-Martin space $\mathcal{H}$ consisting of the Schauder functions. Let $W_i, i \in \mathbb{N}$, be a sequence of independent one-dimensional Wiener processes. Consider

$$dX_t = -\sum_{i=1}^{\infty} \lambda_i \langle S_i, X_t \rangle S_i \, dt + \sum_{i=1}^{\infty} \sqrt{2\lambda_i} \, dW_i(t)S_i$$

$$\equiv AX_t \, dt + B \, d\tilde{W}_t, \quad t \geq 0,$$

where $\langle S_i, X_t \rangle := \int_{u=0}^{1} S_i'(u) \, dX_t(u)$ and $\tilde{W}_t := \sum_{i=1}^{\infty} W_i(t)S_i$, $t > 0$, is a particular Wiener process with values in the Euclidean path space $C_0([0, 1]; \mathbb{R}^d)$. This path space is the Banach space of all continuous $\mathbb{R}^d$ valued functions $f$ on $[0, 1]$ with $f(0) = 0$. 
Solution

Let $G_i, i \in \mathbb{N}$, be the sequence of independent one-dimensional Ornstein-Uhlenbeck processes based on the sequence of independent one-dimensional Wiener processes $W_i, i \in \mathbb{N}$. That is,

$$dG_i(t) = -G_i(t)dt + \sqrt{2} dW_i(t), \quad t \geq 0.$$  

The following process process solves the SPDE,

$$X_t := \sum_{i=1}^{\infty} G_i(\lambda_i t) \cdot S_i, \quad t \geq 0,$$

where the right-hand side converges in the norm of $C_0([0, 1]; \mathbb{R}^d)$. 

Remark

As we will see in the last section of this talk, there are conditions on the initial values $G_i(0), i \in \mathbb{N}$, and the increase of $\lambda_i, i \in \mathbb{N}$, guaranteeing well-definiteness of $X_t, t \geq 0$. 

J. Karlsson, J.-U. Löbus (Linköping)
Solution

Let $G_i, i \in \mathbb{N}$, be the sequence of independent one-dimensional Ornstein-Uhlenbeck processes based on the sequence of independent one-dimensional Wiener processes $W_i, i \in \mathbb{N}$. That is,

$$dG_i(t) = -G_i(t) \, dt + \sqrt{2} \, dW_i(t), \quad t \geq 0.$$ 

The following process process solves the SPDE,

$$X_t := \sum_{i=1}^{\infty} G_i(\lambda_i t) \cdot S_i, \quad t \geq 0,$$

where the right-hand side converges in the norm of $C_0([0, 1]; \mathbb{R}^d)$.

Remark

As we will see in the last section of this talk, there are conditions on the initial values $G_i(0), i \in \mathbb{N}$, and the increase of $\lambda_i, i \in \mathbb{N}$, guaranteeing well-definiteness of $X_t, t \geq 0$. 
Associated generator on $L^2$ w.r.t. Wiener space

Let $z_i := \langle S_i, X_t \rangle \equiv \int_{u=0}^{1} S'_i(u) \, dX_t(u)$. We have

$$AF = \sum_{i=1}^{\infty} \lambda_i \left\{ \partial_{S_i} \partial_{S_i} F - z_i \partial_{S_i} F \right\}$$

on cylindrical functions $F$ specified below.
Associated generator on $L^2$ w.r.t. Wiener space

Let $z_i := \langle S_i, X_t \rangle \equiv \int_{u=0}^{1} S_i'(u) \, dX_t(u)$. We have

$$AF = \sum_{i=1}^{\infty} \lambda_i \{ \partial S_i \partial S_i F - z_i \partial S_i F \}$$

on cylindrical functions $F$ specified below.

Associated Dirichlet form on $L^2$ w.r.t. Wiener space

$$\mathcal{E}(F, G) = \int \sum_{i=1}^{\infty} \lambda_i \langle DF, S_i \rangle_{\mathbb{H}} \langle DG, S_i \rangle_{\mathbb{H}} \, d\nu$$

on cylindrical functions $F$ and $G$ specified below.
Coercive forms

A bilinear form $\mathcal{E} : H \times H \rightarrow \mathbb{R}$ is called coercive if
\[ \mathcal{E}(x, x) \geq C \|x\|_H, \quad \text{for all } x \in D(\mathcal{E}) = H. \]
Coercive forms

A bilinear form $\mathcal{E} : H \times H \to \mathbb{R}$ is called coercive if

$$\mathcal{E}(x, x) \geq C\|x\|_H, \quad \text{for all } x \in D(\mathcal{E}) = H.$$

A coercive closed form is called a Dirichlet form if $u \in D(\mathcal{E})$ implies

(i) $u^+ \land 1 \in D(\mathcal{E}),$
(ii) $\mathcal{E}(u + u^+ \land 1, u - u^+ \land 1) \geq 0,$
(iii) $\mathcal{E}(u - u^+ \land 1, u + u^+ \land 1) \geq 0.$
Coercive forms

A bilinear form $\mathcal{E} : H \times H \to \mathbb{R}$ is called coercive if

$$\mathcal{E}(x, x) \geq C\|x\|_H, \quad \text{for all } x \in D(\mathcal{E}) = H.$$

A coercive closed form is called a Dirichlet form if $u \in D(\mathcal{E})$ implies

(i) $u^+ \wedge 1 \in D(\mathcal{E}),$
(ii) $\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0,$
(iii) $\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0.$

For symmetric forms,

$$\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u, u).$$
In the 1970s Fukushima showed that if a Dirichlet form on a locally compact state space is *regular* one can construct an associated Markov process.
In the 1970s Fukushima showed that if a Dirichlet form on a locally compact state space is *regular* one can construct an associated Markov process.

1992 Ma and Röckner solved the question in an infinite dimensional setting:

\[
\text{Theorem (Z.-M. Ma and M. Röckner 1992)}
\]
In the 1970s Fukushima showed that if a Dirichlet form on a locally compact state space is regular one can construct an associated Markov process.

1992 Ma and Röckner solved the question in an infinite dimensional setting:

**Theorem ([Z.-M. Ma and M. Röckner 1992])**

A Dirichlet form on a Hausdorff topological space is associated with a certain class of Markov processes (\(m\)-special standard) if and only if the form is quasi-regular.
The Dirichlet form

We are concerned with Dirichlet forms of type

\[ E(F, G) = \int \langle DF, ADG \rangle_H \, \varphi \, d\nu \]

\[ = \int \left\langle DF, \sum_{i=1}^{\infty} \lambda_i \langle S_i, DG \rangle_H S_i \right\rangle_H \varphi \, d\nu. \]
The Dirichlet form

We are concerned with Dirichlet forms of type

\[ \mathcal{E}(F, G) = \int \langle DF, ADG \rangle \varphi d\nu \]

\[ = \int \left\langle DF, \sum_{i=1}^{\infty} \lambda_i \langle S_i, DG \rangle \right\rangle \varphi d\nu. \]

Here \( F \) and \( G \) are certain cylindrical functions to be specified below. \( \mathbb{H} \) is the Cameron-Martin space, that is the space of absolutely continuous \( \mathbb{R}^d \)-valued functions \( f \) on \([0, 1]\) with \( f(0) = 0 \). Furthermore, \( \nu \) is the Wiener measure and \( \varphi \) is a non-negative weight function with \( \int \varphi d\nu = 1 \).
Classes of cylindrical functions

We study the form on the set of functions

\[ Z := \left\{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : 0 < s_1 < \cdots < s_k = 1, f \in C^\infty_p \left( (\mathbb{R}^d)^k \right), k \in \mathbb{N} \right\}, \]

where \( f \in C^\infty_p \) means that \( f \) and all its partial derivatives are smooth with polynomial growth, as well as on
Classes of cylindrical functions

We study the form on the set of functions

\[ Z := \left\{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : \right. \]
\[ 0 < s_1 < \cdots < s_k = 1, f \in C^\infty_p \left( (\mathbb{R}^d)^k \right), k \in \mathbb{N} \}, \]

where \( f \in C^\infty_p \) means that \( f \) and all its partial derivatives are smooth with polynomial growth,
Classes of cylindrical functions

We study the form on the set of functions

\[ Z := \{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : \]
\[ 0 < s_1 < \cdots < s_k = 1, f \in C^\infty_p \left( (\mathbb{R}^d)^k \right), k \in \mathbb{N} \}, \]

where \( f \in C^\infty_p \) means that \( f \) and all its partial derivatives are smooth with polynomial growth, as well as on

\[ Y := \{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : \]
\[ F \in Z, s_1, \ldots, s_k \in \left\{ \frac{l}{2^n} : l \in \{1, \ldots, 2^n\} \right\}, n \in \mathbb{N} \}. \]
Classes of cylindrical functions

We study the form on the set of functions

\[ Z := \left\{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : 0 < s_1 < \cdots < s_k = 1, f \in C^\infty_p \left( (\mathbb{R}^d)^k \right), k \in \mathbb{N} \right\}, \]

where \( f \in C^\infty_p \) means that \( f \) and all its partial derivatives are smooth with polynomial growth, as well as on

\[ Y := \left\{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : F \in Z, s_1, \ldots, s_k \in \left\{ \frac{l}{2^n} : l \in \{1, \ldots, 2^n\} \right\}, n \in \mathbb{N} \right\}. \]

Let \( Z_H \) denote the set of all cylindrical functions with values in \( \mathbb{H} \) of the form \( \sum_{i=1}^k F_j h_j \) where \( F_j \in Z, h_j \in H \) and \( k \in \mathbb{N} \).
Closability

Definition

Let $\mathcal{E}$ be a positive definite bilinear form on $H$ with domain $\mathcal{C}$. $(\mathcal{E}, \mathcal{C})$ is called closable on $H$ if for all sequences $u_n \in \mathcal{C}$, $n \in \mathbb{N}$, such that

- $\mathcal{E}(u_n - u_m, u_n - u_m) \xrightarrow{m,n \to \infty} 0$ and
- $u_n \to 0$ in $H$,

we have $\mathcal{E}(u_n, u_n) \to 0$.
Closability

Definition

Let $\mathcal{E}$ be a positive definite bilinear form on $H$ with domain $\mathcal{C}$. $(\mathcal{E}, \mathcal{C})$ is called closable on $H$ if for all sequences $u_n \in \mathcal{C}$, $n \in \mathbb{N}$, such that

- $\mathcal{E}(u_n - u_m, u_n - u_m) \xrightarrow{m,n \to \infty} 0$ and
- $u_n \to 0$ in $H$,

we have $\mathcal{E}(u_n, u_n) \to 0$.

Proposition

A bilinear form is closable if and only if it has a closed extension.
Closability of the gradient $D$ and the divergence $\delta$

Lemma

If

\[
\frac{f}{\varphi} \in L^1(\nu), \quad f \in Z. \tag{1}
\]

Then the operators $D$ and $\delta$ are closable as operators

$D : L^2(\varphi \nu) \supset Z \rightarrow L^2(\varphi \nu; \mathbb{H})$ and $\delta : L^2(\varphi \nu; \mathbb{H}) \supset Z_{\mathbb{H}} \rightarrow L^2(\varphi \nu)$. 
Closability of $D^{-1}$ and $\delta^{-1}$

Equivalence classes

\[
\begin{align*}
\mathbf{x} & \equiv \mathbf{x}(x) := \{x + c \mathbf{1} : c \in \mathbb{R}\}, \\
L^2(\varphi \nu) \ominus 1 & := \{\mathbf{x}(x) : x \in L^2(\varphi \nu)\}, \\
\delta(z) & := \text{The equivalence class of } \delta(z), \ z \in Z_{\mathbb{H}}.
\end{align*}
\]
Closability of $D^{-1}$ and $\delta^{-1}$

Equivalence classes

$$ x \equiv x(x) := \{ x + c1 : c \in \mathbb{R} \}, $$

$$ L^2(\varphi\nu) \ominus 1 := \{ x(x) : x \in L^2(\varphi\nu) \}, $$

$$ \delta(z) := \text{The equivalence class of } \delta(z), \ z \in \mathbb{Z}_{\mathbb{H}}. $$

Lemma

If

$$ f \varphi \in L^1(\nu), \ f \in Z. \quad (1) $$

Then the operators $D^{-1}$ and $\delta^{-1}$ are closable as operators

$$ D^{-1} : L^2(\varphi\nu; \mathbb{H}) \supset \mathbb{Z}^{-1} \equiv \{ Dx : x \in \mathbb{Z} \} \rightarrow L^2(\varphi\nu) \ominus 1 $$

and

$$ \delta^{-1} : L^2(\varphi\nu) \ominus 1 \supset \mathbb{Z}^{-1}_{\mathbb{H}} \equiv \{ \delta(z) : z \in \mathbb{Z}_{\mathbb{H}} \} \rightarrow L^2(\varphi\nu). $$
Closability of the form

**Theorem**

Let \( \varphi \) satisfy (1) and assume that \( A \geq \varepsilon \text{Id} \) for some \( \varepsilon > 0 \).

(a) The form \((\mathcal{E}, Y)\) is closable in \( L^2(\varphi \nu) \). Let \((\mathcal{E}, D_Y(\mathcal{E}))\) denote the closure of \((\mathcal{E}, Y)\) in \( L^2(\varphi \nu) \).

(b) If

\[
\sup_{c_1, c_2, \ldots \in \{0,1\}} \sum_{j=1}^d \sum_{p=0}^\infty \int \lambda_{i_p,j} \varphi \, d\nu \frac{1}{2^p} < \infty,
\]

then \((\mathcal{E}, Z)\) is closable and we let \((\mathcal{E}, D_Z(\mathcal{E}))\) denote the closure of \((\mathcal{E}, Z)\) on \( L^2(\varphi \nu) \). Furthermore \( D_Z(\mathcal{E}) = D_Y(\mathcal{E}) \) under (2).

(c) \( Z \subset D_Y(\mathcal{E}) \) if and only if (2).

(d) If \((\mathcal{E}, Z)\) is closable in \( L^2(\varphi \nu) \) then (2) holds.
The index in (2)

Decoding of the index

\( i_{p,j} \equiv i_{p,j}(c_0, c_1, \ldots) := \begin{cases} 
2^{p-1}(d + j - 1) + 1 + \sum_{q=0}^{p-1} c_q 2^{p-q-1}, & p \geq 1, \\
0, & p = 0,
\end{cases} \)

where \( c_1, \ldots, c_{r-1} \in \{0, 1\} \), \( c_r = 1 \), \( c_0 = c_{r+1} = 0 \) and \( r \geq 2 \).
Decoding of the index

\[ i_{p,j} \equiv i_{p,j}(c_0, c_1, \ldots) := \begin{cases} 2^{p-1}(d + j - 1) + 1 + \sum_{q=0}^{p-1} c_q 2^{p-q-1}, & p \geq 1, \\ j, & p = 0, \end{cases} \]

where \( c_1, \ldots, c_{r-1} \in \{0, 1\}, \; c_r = 1, \; c_0 = c_{r+1} = 0 \) and \( r \geq 2 \)

Remark

If the sequence of eigenvalues is non-decreasing \( 0 < \lambda_1(\gamma) \leq \lambda_2(\gamma), \ldots, \)
then the closability condition (2) simplifies to

\[ \sum_{p=0}^{\infty} \frac{\int \lambda_{d2^p} \varphi d\nu}{2^p} < \infty. \]
(i) An increasing sequence \((F_k)_{k \in \mathbb{N}}\) of closed subsets of \(E\) is called an \(E\)-nest if \(\bigcup_{k \geq 1} D(E)_{F_k}\) is dense in \(D(E)\) w.r.t. \(\| \cdot \|_{\tilde{E}_1}\), where \(D(E)_{F_k}\) denotes \(\{ u \in D(E) : u = 0 \text{ m-a.e. on } E \setminus F_k \}\).
Quasi-regularity – Preliminary definitions

Definition

(i) An increasing sequence \((F_k)_{k \in \mathbb{N}}\) of closed subsets of \(E\) is called an \(\mathcal{E}\)-nest if \(\bigcup_{k \geq 1} D(\mathcal{E})_{F_k}\) is dense in \(D(\mathcal{E})\) w.r.t. \(\| \cdot \|_{\mathcal{E}_1}\), where \(D(\mathcal{E})_{F_k}\) denotes \(\{u \in D(\mathcal{E}) : u = 0\text{ m-a.e. on } E \setminus F_k\}\).

(ii) A subset \(N \subset E\) is called \(\mathcal{E}\)-exceptional if \(N \subset \cap_{k \geq 0} F_k^c\) for some \(\mathcal{E}\)-nest \((F_k)_{k \in \mathbb{N}}\).

(iii) We say that a property holds \(\mathcal{E}\)-quasi-everywhere if it holds everywhere outside some \(\mathcal{E}\)-exceptional set.

Definition

An \(\mathcal{E}\)-quasi-everywhere defined function \(f\) is called \(\mathcal{E}\)-quasi-continuous if there exists an \(\mathcal{E}\)-nest \((F_k)_{k \in \mathbb{N}}\) such that \(f\) is continuous on \((F_k)_{k \in \mathbb{N}}\).
Quasi-regularity – Preliminary definitions

Definition

(i) An increasing sequence \((F_k)_{k \in \mathbb{N}}\) of closed subsets of \(E\) is called an \(\mathcal{E}\)-nest if \(\bigcup_{k \geq 1} D(\mathcal{E})_{F_k}\) is dense in \(D(\mathcal{E})\) w.r.t. \(\| \cdot \|_{\tilde{\mathcal{E}}_1}\), where \(D(\mathcal{E})_{F_k}\) denotes \(\{u \in D(\mathcal{E}) : u = 0 \text{ m-a.e. on } E \setminus F_k\}\).

(ii) A subset \(N \subset E\) is called \(\mathcal{E}\)-exceptional if \(N \subset \cap_{k \geq 0} F_k^c\) for some \(\mathcal{E}\)-nest \((F_k)_{k \in \mathbb{N}}\).

(iii) We say that a property holds \(\mathcal{E}\)-quasi-everywhere if it holds everywhere outside some \(\mathcal{E}\)-exceptional set.

Definition

An \(\mathcal{E}\)-quasi-everywhere defined function \(f\) is called \(\mathcal{E}\)-quasi-continuous if there exists an \(\mathcal{E}\)-nest \((F_k)_{k \in \mathbb{N}}\) such that \(f\) is continuous on \((F_k)_{k \in \mathbb{N}}\).
Quasi-regularity – Preliminary definitions

Definition

(i) An increasing sequence \((F_k)_{k \in \mathbb{N}}\) of closed subsets of \(E\) is called an \(E\)-nest if \(\bigcup_{k \geq 1} D(E)_{F_k}\) is dense in \(D(E)\) w.r.t. \(\| \cdot \|_{\tilde{E}_1}\), where \(D(E)_{F_k}\) denotes \(\{ u \in D(E) : u = 0 \text{ m-a.e. on } E \setminus F_k \}\).

(ii) A subset \(N \subset E\) is called \(E\)-exceptional if \(N \subset \cap_{k \geq 0} F_k^c\) for some \(E\)-nest \((F_k)_{k \in \mathbb{N}}\).

(iii) We say that a property holds \(E\)-quasi-everywhere if it holds everywhere outside some \(E\)-exceptional set.

Definition

An \(E\)-quasi-everywhere defined function \(f\) is called \(E\)-quasi-continuous if there exists an \(E\)-nest \((F_k)_{k \in \mathbb{N}}\) such that \(f\) is continuous on \((F_k)_{k \in \mathbb{N}}\).
Definition of quasi-regularity

Definition

A Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(E, m)\) is called quasi-regular if:

(i) There exists an \(\mathcal{E}\)-nest \((F_k)_{k \in \mathbb{N}}\) consisting of compact sets,

(ii) There exists an \(\tilde{\mathcal{E}}_1^{1/2}\)-dense subset of \(D(\mathcal{E})\) whose elements have \(\mathcal{E}\)-quasi-continuous \(m\)-versions,

(iii) There exist \(u_n \in D(\mathcal{E}), n \in \mathbb{N}\), having \(\mathcal{E}\)-quasi-continuous \(m\)-versions \(\tilde{u}_n, n \in \mathbb{N}\), and an \(\mathcal{E}\)-expectional set \(N \subset E\) such that \(\{\tilde{u}_n : n \in \mathbb{N}\}\) separates the points of \(E \setminus N\).
We consider weight functions $\varphi^+$ and $\varphi^-$ of the form

$$
\varphi^+(\gamma) := \exp \left\{ \int_0^1 \langle b_s(\gamma), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\},
$$

(3)
The weight function $\varphi$

We consider weight functions $\varphi^+$ and $\varphi^-$ of the form

$$
\varphi^+(\gamma) := \exp \left\{ \int_0^1 \langle b_s(\gamma), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\},
$$

and

$$
\varphi^-(\gamma) := \exp \left\{ \int_0^1 \langle -b_s(\gamma), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\},
$$

where $b_s(\gamma)$ is adapted to the natural filtration of the Wiener process.
The weight function $\varphi$

We consider weight functions $\varphi^+$ and $\varphi^-$ of the form

$$\varphi^+(\gamma) := \exp \left\{ \int_0^1 \langle b_s(\gamma), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\}, \quad (3)$$

and

$$\varphi^-(\gamma) := \exp \left\{ \int_0^1 \langle -b_s(\gamma), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\}, \quad (4)$$

where $b_s(\gamma)$ is adapted to the natural filtration of the Wiener process.

Under the condition

$$\frac{f}{\varphi^+} \in L^1(\nu), \quad \text{and} \quad \frac{f}{\varphi^-} \in L^1(\nu), \quad (5)$$

for all $f \in Z$, the earlier closability results hold for $\varphi^+$ as well as for $\varphi^-$. 
The Novikov condition

Lemma

Let \( \varphi^+ \), and \( \varphi^- \) be defined as in (3), and (4). If (5) holds then we have the Novikov condition

\[
E \left[ \exp \left\{ \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\} \right] < \infty.
\]

In particular, \( E[\varphi^+] = E[\varphi^-] = 1 \).
The Novikov condition

Lemma

Let $\varphi^+$, and $\varphi^-$ be defined as in (3), and (4). If (5) holds then we have the Novikov condition

$$E \left[ \exp \left\{ \frac{1}{2} \int_0^1 |b_s(\gamma)|^2 \, ds \right\} \right] < \infty.$$ 

In particular, $E[\varphi^+] = E[\varphi^-] = 1$. 
Theorem

Assume that $A \geq \varepsilon \text{Id}$ for some $\varepsilon > 0$ and suppose (2) and (5). For both, $\varphi = \varphi^+$ as well as $\varphi = \varphi^-$, the closure of

$$\mathcal{E}(F, F) = \sum_{i=1}^{\infty} \int \lambda_i \langle S_i, DF \rangle_{\mathcal{H}}^2 \varphi d\nu, \quad F \in Z,$$

in $L^2(\varphi \nu)$, is quasi-regular.
Let $M$ be a connected geometrically, stochastically complete, and torsion free Riemannian manifold.
Let $M$ be a connected geometrically, stochastically complete, and torsion free Riemannian manifold. We study the form

$$\mathcal{E}(\hat{F}, \hat{G}) = \int \langle \hat{D}\hat{F}, A\hat{D}\hat{G} \rangle_{H} \hat{\varphi} d\hat{\nu}$$

$$= \int \left( \langle \hat{D}\hat{F}, \sum_{i=1}^{\infty} \lambda_{i} \langle S_{i}, \hat{D}\hat{G} \rangle_{H} S_{i} \rangle_{H} \right) \hat{\varphi} d\hat{\nu}$$

(6)

where $\hat{\nu}$ is the Wiener measure on the path space $P_{m_{0}}(M) := \{ \hat{\gamma} \in C([0, 1]; M) : \hat{\gamma}(0) = m_{0} \}$, $m_{0} \in M$ and $\hat{\varphi}$ is a weight function to be specified.
Geometric framework cont.

More precisely let $P_0(\mathbb{R}^d) \equiv \Omega := \{ \gamma \in C([0, 1]; \mathbb{R}^d) : \gamma(0) = 0 \}$, $I : P_0(\mathbb{R}^d) \rightarrow P_{m_0}(M)$ be the Itô map and $\hat{\nu}$ be the image measure of the Wiener measure on $P_0(\mathbb{R}^d)$ under $I$. 

In order to recall the construction of the Itô map let $O(M)$ denote the orthonormal frame bundle with respect to $M$, $\pi$ be the canonical projection $O(M) \rightarrow M$ and $H_1, \ldots, H_d$ be the canonical horizontal vector fields. Choose $r_0 \in O(M)$ such that $\pi(r_0) = m_0$. We introduce $r_\gamma$ as the solution to the Stratonovich SDE

\[
\begin{cases}
\partial r_\gamma(t) = \sum_{i=1}^d H_i(r_\gamma(t)) \partial \gamma_i, \\
  r_\gamma(0) = r_0,
\end{cases}
\]

$\gamma = (\gamma_1, \ldots, \gamma_d) \in P_0(\mathbb{R}^d)$. This defines a.e. a mapping $I : P_0(\mathbb{R}^d) \rightarrow P_{m_0}(M)$ by $I(\gamma)(t) := \pi(r_\gamma(t))$, $\gamma \in P_0(\mathbb{R}^d)$, $t \in [0, 1]$. We also denote by $T_xM$ as the tangent space of $M$ at the point $x \in M$. 

J. Karlsson, J.-U. Löbus (Linköping)
More precisely let \( P_0(\mathbb{R}^d) \equiv \Omega := \{ \gamma \in C([0,1]; \mathbb{R}^d) : \gamma(0) = 0 \} \), \\
\( I : P_0(\mathbb{R}^d) \to P_{m_0}(M) \) be the Itô map and \( \hat{\nu} \) be the image measure of the Wiener measure on \( P_0(\mathbb{R}^d) \) under \( I \). In order to recall the construction of the Itô map let \( O(M) \) denote the orthonormal frame bundle with respect to \( M \), \( \pi \) be the canonical projection \( O(M) \to M \) and \( H_1, \ldots, H_d \) be the canonical horizontal vector fields.
More precisely let $P_0(\mathbb{R}^d) \equiv \Omega := \{ \gamma \in C([0, 1]; \mathbb{R}^d) : \gamma(0) = 0 \}$, $I : P_0(\mathbb{R}^d) \rightarrow P_{m_0}(M)$ be the Itô map and $\hat{\nu}$ be the image measure of the Wiener measure on $P_0(\mathbb{R}^d)$ under $I$. In order to recall the construction of the Itô map let $O(M)$ denote the orthonormal frame bundle with respect to $M$, $\pi$ be the canonical projection $O(M) \rightarrow M$ and $H_1, \ldots, H_d$ be the canonical horizontal vector fields. Choose $r_0 \in O(M)$ such that $\pi(r_0) = m_0$. We introduce $r_{\gamma}$ as the solution to the Stratonovich SDE

$$
\begin{cases}
\partial_r_{\gamma}(t) = \sum_{i=1}^{d} H_i(r_{\gamma}(t)) \partial_{\gamma_i}, & t \in [0, 1], \\
r_{\gamma}(0) = r_0,
\end{cases}
$$

(7)

$\gamma = (\gamma_1, \ldots, \gamma_d) \in P_0(\mathbb{R}^d)$.
Geometric framework cont.

More precisely let $P_0(\mathbb{R}^d) \equiv \Omega := \{ \gamma \in C([0, 1]; \mathbb{R}^d) : \gamma(0) = 0 \}$, $I : P_0(\mathbb{R}^d) \to P_{m_0}(M)$ be the Itô map and $\hat{\nu}$ be the image measure of the Wiener measure on $P_0(\mathbb{R}^d)$ under $I$. In order to recall the construction of the Itô map let $O(M)$ denote the orthonormal frame bundle with respect to $M$, $\pi$ be the canonical projection $O(M) \to M$ and $H_1, \ldots, H_d$ be the canonical horizontal vector fields. Choose $r_0 \in O(M)$ such that $\pi(r_0) = m_0$. We introduce $r_\gamma$ as the solution to the Stratonovich SDE

$$
\begin{aligned}
\begin{cases}
\partial r_\gamma(t) &= \sum_{i=1}^{d} H_i(r_\gamma(t)) \partial \gamma_i, & t \in [0, 1], \\
r_\gamma(0) &= r_0,
\end{cases}
\end{aligned}
$$

(7)

$\gamma = (\gamma_1, \ldots, \gamma_d) \in P_0(\mathbb{R}^d)$. This defines a.e. a mapping $I : P_0(\mathbb{R}^d) \to P_{m_0}(M)$ by $I(\gamma)(t) := \pi(r_\gamma(t)), \gamma \in P_0(\mathbb{R}^d), t \in [0, 1]$, the Itô map. We also denote by $T_x M$ as the tangent space of $M$ at the point $x \in M$. 
Let $K \in C([0, \infty))$ non-negative such that

$$\text{Ric}(X, X) \geq -K(r)|X|^2, \quad X \in T_x M, \; x \in B(m_0, r), \; r > 0, \quad (8)$$

where $B(m_0, r)$ is the geodesic ball at $m_0$ with radius $r$. Let $\rho$ denote the Riemannian distance on $M$ and $\rho_{m_0}(x) := \rho(x, m_0), \; x \in M$. We assume that there are constants $c_1, c_2, r_1 > 0$ such that the following conditions hold,

$$\frac{1}{2} \sqrt{(d-1)K(r)} \leq c_1 r, \quad r \geq r_1, \quad (9)$$

and

$$|\text{Ric}(X, Y)|^p \leq c_2 \exp \left\{ \frac{1}{2} e^{-1 - 2c_1 \rho_{m_0}(x)^2} \right\} \quad (10)$$

for some $p \geq 2$ and $x \in M, \; X, Y \in T_x M, \; |X| = |Y| = 1$. We obtain

$$E \left[ \int_0^1 \|\text{Ric}_{r\gamma}(t)\|^p \, dt \right] < \infty. \quad (11)$$
Closability results

Proposition

Suppose (8) - (11). If for $p$ specified in (10) we have

$$\hat{g} \in L^1(\hat{\nu}), \quad \hat{g} \in L^{\frac{p}{2}}(\hat{\nu}),$$

then $(\mathcal{D}, \hat{Z})$ defined by

$$\mathcal{D}(\hat{F}, \hat{G}) := \int \langle \hat{D}\hat{F}, \hat{D}\hat{G} \rangle_{\mathbb{H}} \hat{\phi} d\hat{\nu}, \quad \hat{F}, \hat{G} \in \hat{Z},$$

is closable on $L^2(\hat{\phi}\hat{\nu})$. 

Corollary

Suppose (8) - (11). Under (12) closability of the form

$$\hat{E}(\hat{F}, \hat{G}) = \int \langle \hat{D}\hat{F}, \hat{D}\hat{G} \rangle_{\mathbb{H}} \hat{\phi} d\hat{\nu},$$

on $L^2(\hat{\phi}\hat{\nu})$ corresponds to that in the non-geometric setting.
Closability results

Proposition

Suppose (8) - (11). If for \( p \) specified in (10) we have

\[ \hat{g} \hat{\phi} \in L^1(\hat{\nu}), \quad \hat{g} \in L^{\frac{p}{2}}(\hat{\nu}), \]  

then \( (\hat{\mathcal{D}}, \hat{\mathcal{Z}}) \) defined by \( \hat{\mathcal{D}}(\hat{F}, \hat{G}) := \int \langle \hat{D}\hat{F}, \hat{D}\hat{G} \rangle_H \hat{\phi} d\hat{\nu}, \ \hat{F}, \hat{G} \in \hat{\mathcal{Z}}, \) is closable on \( L^2(\hat{\phi}\hat{\nu}) \).

Corollary

Suppose (8) - (11). Under (12) closability of the form

\[ \hat{\mathcal{E}}(\hat{F}, \hat{G}) = \int \left\langle \hat{D}\hat{F}, \sum_{i=1}^{\infty} \lambda_i \left\langle S_i, \hat{D}\hat{G} \right\rangle_H S_i \right\rangle_H \hat{\phi} d\hat{\nu}, \]  

on \( L^2(\hat{\phi}\hat{\nu}) \) corresponds to that in the non-geometric setting.
We consider $\hat{\varphi}(\hat{\gamma})$ of the form

$$
\hat{\varphi}(\hat{\gamma}) = \exp \left\{ \int_0^1 \left\langle \hat{V}(\hat{\gamma}_t), d\hat{\gamma}_t \right\rangle_{\mathbf{T}_{\hat{\gamma}(t)}} - \frac{1}{2} \int_0^1 \left| \hat{V}(\hat{\gamma}_t) \right|^2_{\mathbf{T}_{\hat{\gamma}(t)}} dt \right\},
$$

$\hat{\gamma} \in P_{m_0}(M)$. If the weight function $\hat{\varphi}$ satisfies the Novikov condition, it is the Radon–Nikodym derivative with respect to $\hat{\nu}$ of the diffusion measure corresponding to the diffusion process on $M$ with generator $\frac{1}{2} \Delta_M + \hat{V}$, see [M. Capitaine, E. P. Hsu, and M. Ledoux 1997].
Novikov condition in the geometric case

The corresponding conditions to (3), (4) are

\[
\hat{f} \hat{\phi}^+ \in L^1(\hat{\nu}) \text{ and } \hat{f} \hat{\phi}^- \in L^1(\hat{\nu})
\]

for all \( \hat{f} \in \hat{Z} \). We get

\[
\hat{E} \left[ \exp \left\{ \frac{1}{2} \int_0^1 |\hat{V}(\hat{\gamma}_t)|_T^{\hat{\gamma}(t)} dt \right\} \right] < \infty
\]

where \( \hat{E} \) denotes expectation taken with respect to \( \hat{\nu} \).
More geometric conditions

In order to obtain quasi-regularity we assume the Ricci curvature to be bounded from below, i.e. there exists some $c \in \mathbb{R}$, not necessarily non-negative, such that

$$\text{Ric}(X, X) \geq c \|X\|^2 \quad X \in T_x M, \ x \in M. \quad (15)$$
More geometric conditions

In order to obtain quasi-regularity we assume the Ricci curvature to be bounded from below, i.e. there exists some $c \in \mathbb{R}$, not necessarily non-negative, such that

$$\text{Ric}(X, X) \geq c\|X\|^2 \quad X \in T_xM, x \in M. \quad (15)$$

We also sharpen (11) and assume in addition

$$E \left[ \int_0^1 \|\text{Ric}_{r\gamma}(t)\|^p dt \varphi \right] < \infty \quad (16)$$

for some $p > 2$ where we remind of the definition of $\text{Ric}_{r\gamma}(t) : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\langle \text{Ric}_{r\gamma}(t)(a), b \rangle := \text{Ric}(r\gamma(t)a, r\gamma(t)b), \quad a, b \in \mathbb{R}^d.$$
Quasi-regularity theorem

**Theorem**

Suppose the geometric conditions (8) - (11) and (15), (16). Furthermore, assume the following conditions on $\hat{\varphi}$ resp. $\varphi$, (12), (14), and (16). Let $A \geq \varepsilon \Id$ for some $\varepsilon > 0$, and

$$\sup_{c_1, c_2, \ldots \in \{0, 1\}} \sum_{j=1}^{d} \sum_{p=0}^{\infty} \int \lambda_{i,p,j} \hat{\varphi} d\hat{\nu} < \infty.$$  \hspace{1cm} (17)

Then

$$\hat{\mathcal{E}}(\hat{F}, \hat{F}) = \int \left\langle \hat{D}\hat{F}, \sum_{i=1}^{\infty} \lambda_i \left\langle S_i, \hat{D}\hat{F} \right\rangle_{H} S_i \right\rangle_{H} \hat{\varphi} d\hat{\nu}, \quad \hat{F} \in \hat{Z},$$  \hspace{1cm} (18)

is quasi-regular in $L^2(\hat{\varphi} \hat{\nu})$. 
Let $G_i, \; i \in \mathbb{N}$, be a sequence of independent one-dimensional Ornstein-Uhlenbeck processes, i.e. we have

$$dG_i(t) = -G_i(t)dt + \sqrt{2} \, dW_i(t), \quad t \geq 0,$$

for a sequence of independent one-dimensional Wiener processes $W_i, \; i \in \mathbb{N}$. 
Let $G_i$, $i \in \mathbb{N}$, be a sequence of independent one-dimensional Ornstein-Uhlenbeck processes, i.e. we have

$$dG_i(t) = -G_i(t)dt + \sqrt{2} dW_i(t), \quad t \geq 0,$$

for a sequence of independent one-dimensional Wiener processes $W_i$, $i \in \mathbb{N}$.

**Lemma**

Suppose $G_i(0)$ are non-random real numbers such that the sum $\sum_{i=1}^{\infty} |G_i(0)| \cdot S_i$ converges in $C_0([0, 1]; \mathbb{R}^d)$. Then the sum

$$X_t := \sum_{i=1}^{\infty} G_i(\lambda_i t) \cdot S_i$$

converges in the norm of $C_0([0, 1]; \mathbb{R}^d)$ for all $t \geq 0$, almost surely.
Provided that
\[
\sum_{m=0}^{\infty} 2^{-m/2} \cdot \lambda d2^m \cdot \left( \max_{d2^m+1 < i \leq d2^{m+1}} |G_i(0)| + m^{1/2} \right) < \infty, \tag{19}
\]
the sequence of processes \( X_t^{(n)} := \sum_{i=1}^{n} G_i(\lambda i t) \cdot S_i, \ t \geq 0 \), converges in distribution to \( X_t, \ t \geq 0 \), in the space \( C_{C_0([0,1];\mathbb{R}^d)}([0,\infty)) \) of all continuous \( C_0([0,1];\mathbb{R}^d) \)-valued trajectories.
Proposition

Provided that

$$\sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \cdot \lambda d^{2m} \cdot \left( \max_{d^{2m}+1 \leq i \leq d^{2m+1}} |G_i(0)| + m^{\frac{1}{2}} \right) < \infty, \quad (19)$$

the sequence of processes $X_t^{(n)} := \sum_{i=1}^{n} G_i(\lambda_i t) \cdot S_i, \ t \geq 0$, converges in distribution to $X_t, \ t \geq 0$, in the space $C_{C_0([0,1];\mathbb{R}^d)}([0,\infty))$ of all continuous $C_0([0, 1]; \mathbb{R}^d)$-valued trajectories.

Remark

The convergence of the sum $\sum_{i=1}^{\infty} |G_i(0)| \cdot S_i$ in $C_0([0, 1]; \mathbb{R}^d)$ and

$$\lambda_n \leq cn^{\frac{1}{2}-\varepsilon}, \quad n \in \mathbb{N}, \quad \text{for some } c > 0 \text{ and } \varepsilon \in (0, \frac{1}{2})$$

are sufficient for (19).
Scalar quadratic variation of $X$.

We consider $X$ as a process with either $C_0([0, 1]; \mathbb{R}^d)$-valued or $L^1([0, 1]; \mathbb{R}^d)$-valued trajectories. Correspondingly, let $\| \cdot \|$ denote the norm in either $C_0([0, 1]; \mathbb{R}^d)$ or $L^1([0, 1]; \mathbb{R}^d)$. Let $\xi_i, i \in \mathbb{N}$, be a sequence of independent identically distributed standard normal random variables and define

$$\theta := 2E \left[ \left\| \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} \xi_i S_i \right\|^2 \right].$$
Scalar quadratic variation of $X$.

We consider $X$ as a process with either $C_0([0, 1]; \mathbb{R}^d)$-valued or $L^1([0, 1]; \mathbb{R}^d)$-valued trajectories. Correspondingly, let $\| \cdot \|$ denote the norm in either $C_0([0, 1]; \mathbb{R}^d)$ or $L^1([0, 1]; \mathbb{R}^d)$. Let $\xi_i, i \in \mathbb{N}$, be a sequence of independent identically distributed standard normal random variables and define

$$\theta := 2E \left[ \left\| \sum_{i=1}^{\infty} \lambda_i^{1/2} \xi_i S_i \right\|^2 \right].$$

Suppose now that

$$\sum_{m=0}^{\infty} 2^{-m/2} \cdot \lambda_{d2^{m+1}} \cdot \left( \max_{d2^m+1 < i \leq d2^{m+1}} |G_i(0)| + m^{1/2} \right) < \infty \quad (20)$$

and note the similarity to (19). Let $\tau^n = \{0 = t^n_0, t^n_1, \ldots, t^n_{k(n)} = T\}, \ n \in \mathbb{N}$, be an arbitrary sequence of partitions on $[0, T], \ T > 0$ such that $\lim_{n \to \infty} |\tau^n| = 0$. 
Ordinary scalar quadratic variation

**Proposition**

In the sense of uniform convergence in probability (ucp) it holds that

\[ [X]_t := \lim_{n \to \infty} \sum_{j: t_j^n \leq t} \left\| X_{t_j^n} - X_{t_{j-1}^n} \right\|^2 = \theta t, \quad t \geq 0. \]
Ordinary scalar quadratic variation

**Proposition**

In the sense of uniform convergence in probability (ucp) it holds that

\[
[X]_t := \lim_{n \to \infty} \sum_{j: t_j^n \leq t} \| X_{t_j^n} - X_{t_{j-1}^n} \|^2 = \theta t, \quad t \geq 0.
\]

Scalar quadratic variation relative to the stochastic calculus via regularization (s. c. v. r.), cf. [F. Russo and P. Vallois 2007], [C. Di Girolami, G. Fabbri, and F. Russo 2014], and [C. Di Girolami and F. Russo 2014].

**Proposition**

It holds that

\[
\frac{1}{\delta} \int_0^t \| X_{s+\delta} - X_s \|^2 \, ds \xrightarrow{\delta \to 0} \theta t \quad \text{ucp on } t \in [0, \infty).
\]
Tensor quadratic variation of the process $X$

in the Banach space $L^1([0, 1]; \mathbb{R}^d) \otimes \pi L^1([0, 1]; \mathbb{R}^d)$ relative to the stochastic calculus via regularization. That is, we examine

$$[X]_t := \lim_{\delta \to 0} \int_0^t \frac{(X_{u+\delta} - X_u) \otimes (X_{u+\delta} - X_u)}{\delta} \, du$$

in the ucp sense w.r.t. the norm in $L^1([0, 1]; \mathbb{R}^d) \otimes \pi L^1([0, 1]; \mathbb{R}^d)$. 
Tensor quadratic variation of the process $X$

in the Banach space $L^1([0, 1]; \mathbb{R}^d) \otimes_\pi L^1([0, 1]; \mathbb{R}^d)$ relative to the stochastic calculus via regularization. That is, we examine

$$[X]_t \otimes := \lim_{\delta \to 0} \int_0^t \frac{(X_{u+\delta} - X_u) \otimes (X_{u+\delta} - X_u)}{\delta} \, du$$

in the ucp sense w.r.t. the norm in $L^1([0, 1]; \mathbb{R}^d) \otimes_\pi L^1([0, 1]; \mathbb{R}^d)$. For this, let $\xi_i, \ i \in \mathbb{N}$, be a sequence of independent standard normal random variables and define

$$\Theta := 2E \left[ \left( \sum_{i=1}^{\infty} \lambda_{i,}^{1/2} \xi_i S_i \right) \otimes \left( \sum_{i'=1}^{\infty} \lambda_{i'}^{1/2} \xi_{i'} S_{i'} \right) \right].$$
Tensor quadratic variation of the process $X$

in the Banach space $L^1([0, 1]; \mathbb{R}^d) \overset{\otimes}{\otimes} \pi L^1([0, 1]; \mathbb{R}^d)$ relative to the stochastic calculus via regularization. That is, we examine

$$[X]_t \overset{\otimes}{=} \lim_{\delta \to 0} \int_0^t \frac{(X_{u+\delta} - X_u) \otimes (X_{u+\delta} - X_u)}{\delta} \, du$$

in the ucp sense w.r.t. the norm in $L^1([0, 1]; \mathbb{R}^d) \overset{\otimes}{\otimes} \pi L^1([0, 1]; \mathbb{R}^d)$. For this, let $\xi_i, i \in \mathbb{N}$, be a sequence of independent standard normal random variables and define

$$\Theta := 2E \left[ \left( \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} \xi_i S_i \right) \otimes \left( \sum_{i'=1}^{\infty} \lambda_{i'}^{\frac{1}{2}} \xi_{i'} S_{i'} \right) \right].$$

**Proposition**

Provided that we have (20) it holds that

$$[X]_t \overset{\otimes}{=} \Theta t, \quad t \geq 0.$$
Itô formula – Preparations

**Definition, ([C. Di Girolami and F. Russo 2014])**

Let \((X_t)_{t \in [0,T]}\) and \((Y_t)_{t \in [0,T]}\) be continuous \(B\)-valued, respectively \(B^*\)-valued stochastic processes. The *forward integral of \(Y\) with respect to \(X\)* denoted by \(\int_0^t B^* \langle Y_s, dX_s \rangle_B\) is defined as the limit

\[
\int_0^t B^* \langle Y_s, dX_s \rangle_B := \lim_{\varepsilon \to 0} \int_0^t B^* \left\langle Y_s, \frac{X_{s+\varepsilon} - X_s}{\varepsilon} \right\rangle_B
\]

in probability.
Itô formula – Preparations

Definition, ([C. Di Girolami and F. Russo 2014])

Let \((X_t)_{t \in [0,T]}\) and \((Y_t)_{t \in [0,T]}\) be continuous \(B\)-valued, respectively \(B^*\)-valued stochastic processes. The *forward integral of \(Y\) with respect to \(X\)* denoted by \(\int_0^t B^*\langle Y_s, dX_s \rangle_B\) is defined as the limit

\[
\int_0^t B^*\langle Y_s, dX_s \rangle_B := \lim_{\varepsilon \to 0} \int_0^t B^*\langle Y_s, \frac{X_{s+\varepsilon} - X_s}{\varepsilon} \rangle_B
\]

in probability.

Definition

Let \(F\) be a mapping \(F : [0, T] \times B \to \mathbb{R}\). We say that \(F\) is of *Fréchet class \(C_{1,2}^1\)* (in symbols \(F \in C_{1,2}^1\)) if \(F\) is one time continuously Fréchet differentiable and two times continuously Fréchet differentiable in the second argument.
Itô formula – An application of the s. c. v. r.

**Theorem**

Let $F \in C^{1,2}$. We have

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial}{\partial s} F(s, X_s) \, ds + \int_0^t B^* \langle DF(s, X_s), dX_s \rangle_B$$

$$+ \frac{1}{2} \int_0^t (B \otimes \pi B)^* \langle D^2 F(s, X_s), \Theta \rangle_{(B \otimes \pi B)^{**}} \, ds$$

where $B = L^1([0, 1]; \mathbb{R}^d)$. 
Theorem

Let $F \in C^{1,2}$. We have

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial}{\partial s} F(s, X_s) \, ds + \int_0^t B^* \langle DF(s, X_s), dX_s \rangle_B$$

$$+ \frac{1}{2} \int_0^t (B \otimes \pi B)^* \langle D^2 F(s, X_s), \Theta \rangle_{(B \otimes \pi B)^*} \, ds$$

where $B = L^1([0, 1]; \mathbb{R}^d)$.

We also verify that, for a certain class of cylindrical functions $F$, the expression on the right hand side takes the well known form of the finite dimensional Itô formula.
corresponding to identification of span \( \{S_1, \ldots, S_k\} \) with \( \mathbb{R}^k \). Note that 
\[
\langle S_i, X_s \rangle = G(\lambda_is).
\]

**Proposition**

For \( F(s, X_s) = f(s; \langle S_1, X_s \rangle, \ldots, \langle S_k, X_s \rangle) \), \( f \in C^\infty_0 (\mathbb{R}^{k+1}) \), \( s \geq 0 \), the following Itô formula holds.

\[
F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial}{\partial s} F(s, X_s) \, ds \\
+ \sum_{i=1}^k \int_0^t \frac{\partial}{\partial x_i} f(s; G_1(\lambda_1 s), \ldots, G_k(\lambda_k s)) \, d_s G_i(\lambda_i s) \\
+ \sum_{i=1}^k \int_0^t \lambda_i \frac{\partial^2}{\partial x_i^2} f(s; G_1(\lambda_1 s), \ldots, G_k(\lambda_k s)) \, ds.
\]
References on the class of stochastic processes

X. Chen and B. Wu.
Functional inequality on path space over a non-compact Riemannian manifold.

J. Karlsson and J.-U. Löbus.
A class of infinite dimensional stochastic processes.

J.-U. Löbus.
A class of processes on the path space over a compact Riemannian manifold with unbounded diffusion.

F.-Y. Wang and B. Wu.
Quasi-regular Dirichlet forms on Riemannian path & loop spaces.
References on stochastic differential geometry

M. Capitaine, E. P. Hsu, and M. Ledoux.
Martingale representation and a simple proof of logarithmic sobolev inequalities on path spaces.

C. Houdr´e and N. Privault.
A concentration inequality on Riemannian path space.

E. P. Hsu and C. Ouyang.
Quasi-invariance of the Wiener measure on the path space over a complete Riemannian manifold.
C. Di Girolami, G. Fabbri, and F. Russo.
The covariation for Banach space valued processes and applications.

C. Di Girolami and F. Russo.
Generalized covariation for Banach space valued processes, Itô formula and applications.

J. Karlsson and J.-U. Löbus.
Infinite dimensional Ornstein-Uhlenbeck processes with unbounded diffusion – Approximation, quadratic variation, and Itô formula.

F. Russo and P. Vallois.
Elements of stochastic calculus via regularization.
Thanks to organizers and participants!