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Bessel process killed upon leaving a half-line

Joint work with Jacek Małecki.

In this talk we study the Bessel process $R_t^{(\mu)}$ with index $\mu \in \mathbf{R}$, starting from $x > 0$. The transition probability density (with respect to the reference measure $m^{(\mu)}(dy) = y^{2\mu+1}dy$) of the process is expressed by the modified Bessel function of the first kind in the following way

$$p^{(\mu)}(t,x,y) = \frac{1}{t}(xy)^{-\mu}\exp\left(-\frac{x^2+y^2}{2t}\right)I_{|\mu|}\left(\frac{xy}{t}\right), \qquad x,y>0,\ t>0.$$

The transition density function of this process killed when it reaches a positive level a (where $x > a$) is given (with respect to $m^{(\mu)}(dy)$) by the Hunt formula

$$p_a^{(\mu)}(t,x,y) = p^{(\mu)}(t,x,y) - \int_0^t p^{(\mu)}(t-s,a,y)q_{x,a}^{(\mu)}(s)ds, \qquad x,y>a,\ t>0,$$

where $q_{x,a}^{(\mu)}(s)$ is the density of the first hitting time $T_a^{(\mu)} = \inf\{t>0:\ R_t^{(\mu)} = a\}$. Our purpose is to derive sharp two-sided estimates of $p_a^{(\mu)}(t,x,y)$. The main result is given in the following

Theorem 1 (KB, J. Małecki, 2015)**.** *Let $\mu \neq 0$ and $a > 0$. Then for every $x, y > a$ and $t > 0$ we have*

$$p_a^{(\mu)}(t,x,y) \overset{\mu}{\approx} \left(1 \wedge \frac{(x-a)(y-a)}{t}\right)\frac{1}{\sqrt{t}}(xy)^{-\mu-1/2} \times \\ \times \exp\left(-\frac{(x-y)^2}{2t}\right)\left(1\wedge\frac{xy}{t}\right)^{\mu-1/2}.$$

Moreover if $\mu = 0$ and $a > 0$, then we have for every $x, y > a$ and $t > 0$ that

$$p_a^{(0)}(t,x,y) \approx \ln(x/a)\ln(y/a)\left(\ln\frac{3t}{ax+a\sqrt{t}}\ln\frac{3t}{ay+a\sqrt{t}}\right)^{-1}\frac{1}{t}\exp\left(-\frac{x^2+y^2}{2t}\right)$$

when $xy \le t$, and

$$p_a^{(0)}(t,x,y) \approx \left(1\wedge\frac{(x-a)(y-a)}{t}\right)\frac{1}{\sqrt{xyt}}\exp\left(-\frac{(x-y)^2}{2t}\right)$$

when $xy > t$.

Here $f(t,x,y) \overset{\mu}{\approx} g(t,x,y)$ means that there exist positive constants $c_1^{(\mu)}$ and $c_2^{(\mu)}$ depending only on the index μ such that $c_1^{(\mu)} \leq f/g \leq c_2^{(\mu)}$ for every $x, y > a$ and $t > 0$. We write $\approx$ instead of $\overset{\mu}{\approx}$ if these constants do not depend on any parameter.

Notice that the constants appearing in the exponential terms in the upper and lower estimates are the same and consequently the exponential behaviour of the density is very precise. We will also discuss our results in context of classical estimates of Dirichlet heat kernel for Laplacian and estimates of Fourier–Bessel heat kernel.

This presentation is based on joint work with Jacek Małecki ([1] and [2]).

References

[1] K. Bogus, J. Małecki, *Sharp estimates of transition probability density for Bessel process in half-line*, Potential Analysis, DOI 10.1007/s11118-015-9461-x, p.1-22, 2015.

[2] K. Bogus, J. Małecki, *Heat kernel estimates for the Bessel differential operator in half-line*, preprint, arXiv:1501.02618, p. 1-10, 2015.