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Nonlocal Schrödinger operators with potentials decaying at infinity

Joint work with József Lőrinczi.

We study the spatial decay of eigenfunctions of non-local Schrödinger operators

$$H = -L + V$$

based on generators L of symmetric jump-paring Lévy processes X_t with Kato-class (with respect to X_t) potentials V decaying at infinity (i.e. $V(x) \to 0$ as $|x| \to \infty$). This class of processes has the property that the intensity of single large jumps dominates the intensity of all multiple large jumps.

For confining potentials (i.e. $V(x) \to \infty$ as $|x| \to \infty$), a similar problem has been recently considered in [1]. Taking a sufficiently regular Kato-class confining potential V, we proved that for large arguments the ground state of H behaves like

$$\varphi_0(x) \asymp \frac{\nu(x)}{V(x)}, \tag{1}$$

where ν is the density of the Lévy measure of X_t (Lévy density, in short). This gives a neat account of the separate contributions of the unperturbed process and of the perturbation into the decay. For general confining Kato-class potentials, (1) has the generic form

$$\varphi_0(x) \asymp \nu(x)\, \Lambda_V(x), \tag{2}$$

where $\Lambda_V(x)$ is the mean exit time from a unit ball centered at the starting point x of the process under the potential V. This formula gives a probabilistic interpretation to the decay of ground states, from which the estimate (1) is also derived. We also proved that the other eigenfunctions φ satisfy

$$|\varphi(x)| \leq C_{\varphi,V,X}\, \varphi_0(x), \quad x \in \mathbf{R}^d.$$

For decaying potentials, we find that the decay rates of eigenfunctions depend on the process via specific preference rates in particular jump scenarios, and depend on the potential through the distance of the corresponding eigenvalue from the edge of the continuous spectrum. We prove that the conditions of the jump-paring class imply that for all eigenvalues the corresponding positive eigenfunctions decay at most as rapidly as the Lévy density ν. This condition is sharp in the sense that if the jump-paring property fails to hold, then eigenfunction decay becomes slower than the decay of ν. We furthermore prove that under reasonable conditions the

Lévy density also governs the upper bounds of eigenfunctions, and a ground state is comparable to it by two-sided bounds; this holds for sufficiently low-lying negative eigenvalues for all jump-paring processes, and for all negative eigenvalues under mild extra conditions on the processes. We thus show that a counterpart of (1)-(2) holds. Indeed, for a potential V decaying to zero the perturbed processes behave far out like free processes, thus we have $\Lambda_V(x) \asymp \text{const}$ and (2) reduces to $\varphi_0 \asymp \nu$.

As an interesting consequence of our results, we identify a sharp regime change (phase transition) in the decay of eigenfunctions as the Lévy density ν is varied from sub-exponential to exponential order, and dependent on the location of the eigenvalue, in the sense that through the transition Lévy density-driven decay becomes slower than the rate of ν. We also give a heuristic explanation to this property. Our approach is based on path integration and probabilistic potential theory techniques, and all results are illustrated by specific examples.

The talk is based on joint paper [2] with József Lőrinczi (Loughborough).

References

[1] K. Kaleta, J. Lőrinczi, *Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes*, Ann. Probab., to appear, 2015.

[2] K. Kaleta, J. Lőrinczi, *Fall-off of eigenfunctions for nonlocal Schrödinger operators with decaying potentials*, preprint, 2015.