Local fluctuations of critical Mandelbrot cascades

Konrad Kolesko
joint with
D. Buraczewski and P. Dyszewski
Będlewo, 4 May, 2015
For given random variables $X_1, X_2$ s.t. $\mathbb{E}e^{-X_1} + e^{-X_2} = 1$ we are interested in random measures $\mu$ on $[0,1)$ satisfying self similar property:

$$\mu(B) = e^{-X_1} \mu_1(2(B \cap [0,1/2])) + e^{-X_2} \mu_2(2(B \cap [1/2,1) - 1)),$$

where $\mu_1 \perp \mu_2 \perp X_1, X_2$ and $\mathcal{L}\mu = \mathcal{L}\mu_1 = \mathcal{L}\mu_2$.

**Goal:** Understand local properties of $\mu$.
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For any metric space \((X, d)\) and \(A \subset X\) by \(\dim_H^d(A)\) we denote its Hausdorff dimension:

\[
\dim_H(A) = \inf \left\{ s : \inf \{ \text{\{\(B_i\}\)-cover} A \sum_i \text{diam}(B_i)^s = 0 \} \right\}
\]

For any finite measure \(\mu\) by \(\dim(\mu)\) we denote its dimension:

\[
\dim(\mu) = \inf \{ \dim_H(A) : \mu(A^c) = 0 \}
\]

For any Borel \(A\) such that \(\mu(A) > 0\) the following holds:

\[
\inf_{x \in A} \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \leq \dim_H(A) \leq \sup_{x \in A} \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}
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Hausdorff dimension

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\]
Define $\psi(t) = \log_2 \mathbb{E}(e^{-tX_1} + e^{-tX_2})$. By assumption $\psi(0) = 1$, $\psi(1) = 0$, $\psi \nearrow \infty$.

There are three cases:

1. **Subcritical**: There is a root of $\psi$ in $(1, \infty)$, $\psi'(1) < 0$

2. **Critical**: $1$ is the only root of $\psi$, $\psi'(1) = 0$

3. **Supercritical**: There is a root of $\psi$ in $(0, 1)$, $\psi'(1) > 0$
Subcritical case $\psi'(1) = -m < 0$

**KPZ relation - Benjamini, Schramm**

For $d_\mu(x, y) = \mu([x, y])$ then for any $A$

$$\dim(A) = \dim_{d_\mu}(A) - \psi(\dim_{d_\mu}(A))$$

**Multifractal analysis - Holley, Waymire; Molchan**

For $E_\mu(\gamma) = \left\{ x : \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \gamma \right\}$ and $\tau(s) = -1 - \psi(s)$

$$\dim(E_\mu(\gamma)) = \tau^*(\gamma)$$

**Pointwise fluctuation - Liu**

$$2^{-m+(1-\delta)\sqrt{2\sigma^2 n \log \log n}} \leq \mu(B(x, 2^{-n})) \leq 2^{-m+(1+\delta)\sqrt{2\sigma^2 n \log \log n}}$$
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Barral, Rhodes, Vargas

When $\psi'(1) > 0$ then $\mu$ is purely atomic

Barral, Kupiainen, Nikula, Saksman, Webb

If $\psi'(1) = 0$ then the random measure $\mu$ almost surely has no atoms. Moreover for any $k$ and $\delta > 0$

1. $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{6 \log 2} \sqrt{n(\log n + (1/3+\delta) \log \log n)}}$ for sufficiently large $n$
2. $\mu(B(x, 2^{-n})) \geq e^{-(\sqrt{2 \log 2}+\delta) \sqrt{n \log n}}$ i.o.
3. $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large $n$ and for $\mu$-a.e. $x$. 
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- $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large $n$ and for $\mu$-a.e. $x$. 
We may identify dyadic intervals with a vertices of a binary rooted tree $\tau$ and any point $x \in [0, 1)$ with $\theta \in \partial \tau$.

We write

$$B(v) = \{ \theta \in \partial \tau : v \text{ is in the geodesic between the root and } \theta \}.$$ 

For a random measure $\mu_\omega$ with a distribution $\mathbb{P}$ we are interested in a pointwise estimates of $\mu_\omega(B(\theta_n))$ on the enlarged measure space $\tilde{\mathbb{P}}(d\omega, d\theta) := \mathbb{P}(d\omega)\mu_\omega(d\theta)$ i.e. we are looking for deterministic $\phi$ s.t.

$$\mu_\omega(B(\theta_n)) \leq \phi(n) \text{ for large } n$$

for $\tilde{\mathbb{P}}$-almost all $(\omega, \theta)$.

$\tilde{\mathbb{P}}$ can be replaced by $\hat{\mathbb{P}}(d\omega, d\theta) = \mathbb{P}(d\omega)\bar{\mu}_\omega(d\theta)$, where $\bar{\mu}_\omega$ is the normalized $\mu_\omega$. 
We may identify dyadic intervals with vertices of a binary rooted tree $\tau$ and any point $x \in [0, 1)$ with $\theta \in \partial\tau$. We write

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We labeled all vertices of the binary rooted tree $\tau$ by independent copies of $X_1$. For any $v \in \tau$ we denote $X(v)$ the sum of random variables along the path between the root and $v$.

The process $$\{X(v)\}_{v \in \tau}$$

is called branching random walk (BRW). We may identify BRW with a random labeled tree.

Ω-set of labeled binary trees and $\mathbb{P}$ is a measure on $\Omega$ i.e.

$$\mathbb{P}(T \in d\omega) = \mathbb{P}(d\omega)$$

When $\psi'(1) \leq 0$ then the random measure $\mu$ is measurable with respect to $\sigma(T)$. 
Since \( E \sum_{|v|=1} e^{-X(v)} = 1 \) and \( E \sum_{|v|=1} X(v)e^{-X(v)} = 0 \) the equation

\[
E f(Y) := E \sum_{|v|=1} f(X(v)) e^{-X(v)}
\]
defines distribution of a driftless r.v. \( Y \)

Let \( h \) be a harmonic function on some set \( D \), 
\( V_n = Y_1 + \cdots + Y_n \) and \( \sigma = \min\{k : s + V_k \notin D\} \). Then the process \( h(s + V_{\min(n,\sigma)}) \) is a martingale.

Let \( \tau = \{w : s + X(w) \notin D \text{ for the first time}\} \), \( v_\tau = \min(v, \tau) \). Then

\[
W_n^s = \sum_{|v|=n} h(s + X(v_\tau)) e^{-X(v_\tau)}
\]
is a martingale.
Some natural martingales

- \( h \equiv 1 \): \( W_n = \sum_{|v|=n} e^{-X(v)} \)
- \( h = x \): \( D_n = \sum_{|v|=n} X(v) e^{-X(v)} \)
- \( D = [0, \infty) \), \( h(x) \approx x \vee 0 \):
  \[ W^s_n = \sum_{|v|=n} h(s + X(v)) 1_{s + X(v') > 0, \text{ for } v' \leq v} e^{-X(v)} \]

\( W^s_n \to W^s \) \( \mathbb{P} \)-a.s. and \( L^1 \)

For any \( s \in D \) define

\[ \mathbb{P}^s := \frac{1}{h(s)} W^s \cdot \mathbb{P} \]
$P^s \Rightarrow \infty$

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Spinal decomposition of $\mathbb{P}^s$
Spinal decomposition of $\mathcal{P}^s$
Spinal decomposition of $P^s$
Spinal decomposition of $\mathcal{P}_s$:

\[ e^{-X_1^1} h(s + X_1^1) \]

\[ e^{-X_2^1} h(s + X_2^1) \]
Spinal decomposition of $\mathbb{P}_s$

$$S_1 = s + X_2^1$$

$$S_0 = s$$
Spinal decomposition of $\mathbb{P}_s$

$X_1^2$ $X_2^2$

BRW

$\infty$

$S_1$

$S_0$

$0$
Spinal decomposition of $P_s$

\[ X_1^2 e^{-X_1^2 h(S_1 + X_1^2)} \]

\[ X_2^2 e^{-X_2^2 h(S_1 + X_2^2)} \]
Spinal decomposition of $\mathbb{P}^s$
Spinal decomposition of $\mathbb{P}^s$

 Diagram showing branching processes with labels $X^3_1$ and $X^3_2$.
Spinal decomposition of $\mathbb{P}^s$

$X_1^3 e^{-X_1^3} h(S_2 + X_1^3) \quad X_2^3 e^{-X_2^3} h(S_2 + X_2^3)$
Spinal decomposition of $\mathbb{P}^s$

\[ S_3 = S_2 + X_3^1 \]

\[ S_2 = S_1 + X_2^1 \]

\[ S_1 = S_0 + X_1^1 \]

\[ S_0 = \epsilon \]

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Spinal decomposition of $\mathbb{P}^s$
Spinal decomposition of $P^s$

Random tree $T^s$ with a distinguished ray $\Theta \in T^s$
Spinal decomposition of $\mathbb{P}^s$:

$$\mathbb{P}(T^s \in d\omega) = \mathbb{P}^s(d\omega)$$
\[ \hat{\mathbb{P}}_s(d\omega, d\theta) := \mathbb{P}(T_s \in d\omega, \Theta \in d\theta) \]
For any labeled tree $\omega$ (resp. with distinguished ray $\theta$) by $\omega_n$ (resp. $\theta_n$) we denote the restriction up to level $n$. The random variable $(T^s_n, \Theta_n)$ can be constructed in two steps:

1. Choose tree $T^s_n$ with according to the distribution $\mathbb{P}^s$

2. Pick up one of the vertex $v$ from $n$-th level with distribution proportional to

\[
\frac{1}{W^s_n} \cdot h(s + X(v))1_{[s+X(v')>0, \text{ for } v' \leq v]} e^{-X(v)}
\]

i.e.

\[
\frac{1}{W^s_n} \cdot \frac{1}{W^s_n} \cdot h(s + X(v))1_{[s+X(v')>0, \text{ for } v' \leq v]} e^{-X(v)}.
\]

Set $\Theta_n$ to be ray $ov$. 
For $s$ large enough and fixed tree $\omega \in \text{supp}(\mathbb{P})$

\[
\mu_\omega(B(u)) = \lim_{n \to \infty} \sum_{v > u, |v| = n} X(v) e^{-X(v)}
\]

\[
= \lim_{n \to \infty} \sum_{v > u, |v| = n} (s + X(v)) e^{-X(v)}
\]

\[
\approx \lim_{n \to \infty} \sum_{v > u, |v| = n} h(s + X(v)) e^{-X(v)}
\]

\[
\lim_{n \to \infty} \sum_{v > u, |v| = n} h(s + X(v)) 1_{[s+X(v') > 0, \text{for } v' \leq v]} e^{-X(v)}
\]

\[
= W^s(\omega) \cdot \mathbb{P}(\Theta \in B(u) | \mathcal{T}^s = \omega)
\]

In particular $\mathbb{P}(\Theta \in \cdot | \mathcal{T}^s = \omega) \ll \mu_\omega$
\[ P(\Theta \in \cdot | T^s = \omega) \ll \mu_\omega \]

For any \( A \subset T^* \)-set of trees with distinguished ray, such that \( \hat{P}^s(A) = 1 \)

\[
\hat{P}^s(A) = \int \int 1_{(\omega,\theta) \in A} P(\Theta \in \theta | T^s = \omega) P^s(d\omega)
\]

\[
\leq \int \int 1_{(\omega,\theta) \in A} \mu(\theta) P^s(d\omega)
\]

\[
\leq \int \int 1_{(\omega,\theta) \in A} \mu(\theta) P(d\omega) + \epsilon = \hat{P}(A) + \epsilon
\]
Probabilistic interpretation of $\mu$

$$e^{-(S_0 - s)}$$

$$e^{-(S_1 - s)}$$

$$e^{-(S_2 - s)}$$

$C_0$

$R_0$

$C_1$

$C_2$

$R_1$

$R_2$
Under $\widehat{\mathbb{P}}^s$ the sequence $\mu_\omega(B(\theta_n))$ is a random variable $\mu_{T^s}(B(\Theta_n))$ which has the same law as

$$\sum_{k \geq n} e^{-(S_k-s)} C_k R_k,$$

where $S_k$ is a random walk starting form $s$ conditioned to to stay positive (discrete version of a Bessel process), $R_k$ independent random variables.
We are looking for LIL for \( \sum_{k \geq n} e^{-S_k} C_k R_k \),

**Motto’59; Hambly, Kersting, Kyprianou 2003**

- \( \int_{0}^{\infty} \frac{\psi(t)}{t} dt < \infty \) iff \( S_n > \sqrt{n\psi(n)} \) eventually
- \( \lim\sup_n \frac{S_n}{\sqrt{2n\sigma^2 \log \log n}} = 1 \)

\[
\sqrt{n\psi(n)} < S_n < (1 + \delta) \sqrt{2n\sigma^2 \log \log n}
\]

\( \mathbb{P}^s\)-a.s. for \( n \) sufficiently large

**Kyprianou**

- \( 1 - \mathbb{E}e^{-tR} \sim tL(t) \)

In particular, by Borel-Cantelli lemma, \( R_n < n^2 \) for all but finitely many \( n \).
We are looking for LIL for $\sum_{k\geq n} e^{-S_k} R_k$.

**Motto’59; Hambly, Kersting, Kyprianou 2003**

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In particular, by Borel-Cantelli lemma, \( R_n < n^2 \) for all but finitely many \( n \).
Take any $\psi$ such that $\int_{\infty}^{\infty} \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k \geq n} e^{-S_k} R_k$$

is eventually bounded by

$$\sum_{k \geq n} e^{-\sqrt{k} \psi(k)} k^2.$$

On the other hand, if $\int_{\infty}^{\infty} \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n} \psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n} \psi(n)}.$$
Take any $\psi$ such that $\int_0^\infty \frac{\psi(t)dt}{t} < \infty$. We have that

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is eventually bounded by

$$\sum_{k \geq \sqrt{n\psi(n)}} e^{-k} \cdot polynomial(k).$$

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$$\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n\psi(n)}} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n\psi(n)}}$$
Upper bound

Take any \( \psi \) such that \( \int_{\infty}^{\infty} \frac{\psi(t)dt}{t} < \infty \). We have that

\[
\sum_{k \geq n} e^{-S_k} R_k
\]

is eventually bounded by

\[
\sum_{k \geq \sqrt{n}\psi(n)} e^{-k}.
\]

On the other hand, if \( \int_{\infty}^{\infty} \frac{\psi(t)dt}{t} = \infty \) then

\[
\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n}\psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n}\psi(n)}
\]
Take any $\psi$ such that $\int_0^\infty \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k \geq n} e^{-S_k} R_k$$

is eventually bounded by

$$e^{-\sqrt{n\psi(n)}}.$$

On the other hand, if $\int_0^\infty \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n\psi(n)}} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n\psi(n)}}$$
Take $q > 1$ and by $N(n) := q^{\lceil \log q \rceil}$. Borel-Cantelli lemma’s implies that for sufficiently large $n$, $\sup_{q^n < k \leq q^{n+1}} R_k \geq \delta_0$.

$$\sum_{k \geq n} e^{-S_k} R_k \geq \sum_{k \geq n} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k$$

$$\geq \sum_{k=N(n)} e^{-(1+\delta)\sqrt{2qN(n)\sigma^2 \log \log (qN(n))}} R_k$$

$$\geq \delta_0 e^{-(1+\delta)\sqrt{2q^2 n\sigma^2 \log \log (q^2 n)}}$$

$$\geq e^{-(1+2\delta)\sqrt{2n\sigma^2 \log \log n}}$$

for some $q$. 
Take $q > 1$ and by $N(n) := q^{\lceil \log_q \rceil}$. Borel-Cantelli lemma’s implies that for sufficiently large $n$, \( \sup_{q^n < k \leq q^{n+1}} R_k \geq \delta_0 \).

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\]

\[
\geq \sum_{k= N(n)}^{qN(n)} e^{-(1+\delta) \sqrt{2k\sigma^2 \log \log k}} R_k
\]

\[
\geq \delta_0 e^{-(1+\delta) \sqrt{2qN(n)\sigma^2 \log \log (qN(n))}}
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\geq e^{-(1+2\delta) \sqrt{2n\sigma^2 \log \log n}},
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\[
\sum_{k \geq n} e^{-S_k} R_k \geq \sum_{k \geq n} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k \\
\geq \sum_{k=N(n)}^{qN(n)} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k \\
\geq \delta_0 e^{-(1+\delta)\sqrt{2qN(n)\sigma^2 \log \log (qN(n))}} \\
\geq \delta_0 e^{-(1+\delta)\sqrt{2q^2n\sigma^2 \log \log (q^2n)}} \\
\geq e^{-(1+2\delta)\sqrt{2n\sigma^2 \log \log n}},
\]

for some $q$. 
**Theorem (BDK)**

Let $k \in \mathbb{N}$ and $\delta > 0$. Then for $\mathbb{P}$-a.e. labeled tree $\omega$, for $\mu_\omega$-a.e. $\theta \in \partial^* \mathbb{T}$ and sufficiently large $n$ we have

\[
\mu_\omega(B(\theta_n)) \geq \exp \left( -(1 + \delta) \sqrt{2\sigma^2 n \log \log n} \right)
\]

\[
\mu_\omega(B(\theta_n)) \leq \exp \left( \frac{-\sqrt{n}}{\prod_{i=1}^{k} \log(i) n \left( \log(k+1) n \right)^2} \right)
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