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Stirring two grains of sand

The word "stirring" in the title of the talk refers to a random change in a system of many bodies that is caused by a single agent that moves continuously and acts locally. This is in contrast to those stochastic flows where different parts of the moving medium are simultaneously "pushed" by different (although possibly correlated) random noises. In everyday life, stirring typically refers to activities such as stirring coffee in a cup or stirring paint in a bucket. In these situations, stirring the medium with a spoon or a stick causes the bulk of the liquid to move (in a circular fashion). Our model is closer to stirring sand in a sandbox with a stick. In this situation, sand grains are displaced locally and there is no overall motion of the bulk of the sand mass.

Stirring sand in a sandbox provided motivation for this project but our model is a simplification of the reality in (at least) two significant ways. First, we will consider only two "sand grains" represented by balls. This seems to be the crucial step in the analysis of the motion of many "sand grains" (see the remarks on [1] below). Second, the stirring agent will be represented by an infinitely small particle performing Brownian motion. One may consider our results as a first step towards a more realistic model.

In our model, the stirring agent, represented by Brownian motion, is not affected by the motion of "sand grains." The two sand grains (balls) remain motionless except when they are pushed by the Brownian particle aside, when its trajectory hits their surfaces.

The problem that we will investigate is that of the evolution of the vector between the centers of the two balls. It is natural to guess that the motion of a each ball should be similar to that of Brownian motion on the local time scale. The crux of the problem is that the directions of the push that the balls receive from the Brownian particle are not independent. Therefore, even if the guess about the motion of a single ball is correct, that does not immediately imply that the limit distribution for the pair of the balls is a pair of independent Brownian motions. We will prove that this is in fact true and we will express this idea in two different ways, to be described below. The main technical challenge of the paper is to estimate the magnitude of the dependence between motions of the two balls.

We will separately prove the invariance principle for a single ball pushed by Brownian motion in dimension 2 in the whole space $\mathbb{R}^2$. This is meaningful because Brownian motion is recurrent in two dimensions so it will keep pushing the ball forever. We consider this simplified question separately to present a more or less straightforward proof. Many technical details obscure this part of the argument in the case of two balls or in higher dimensions.

In dimensions 3 and higher, the two balls and Brownian motion will be located in a torus because Brownian motion is transient in these dimensions (but the theorem will cover the two dimensional case as well). First, we will prove an invariance principle on the local time scale for the centers of the two balls. The limiting process is a pair of two independent Brownian motions. Next we will show that the rescaled stationary distributions for the two balls in a torus of diameter r converge to the product of the stationary (and hence uniform) distributions for the individual balls as $r \to \infty$. We will explain why the latter theorem does not immediately follow from the former.

The present paper is a part of a larger project. Our present model is "almost" equivalent to the model in which a ball with the center moving as a Brownian motion pushes two point-like particles. The equivalence is not complete because the two balls in our model cannot intersect (by assumption) and hence their centers are always at least two units apart. In the other model, the two point-like particles can come arbitrarily close. Their motion was partly analyzed in [1], where it was proved that the distance between the two particles does not converge to 0 in a three dimensional torus. This is very close to proving recurrence for the two-particle process. The main results of the present article show, more or less, that the particles are independent on the large scale. Only one element of the program initiated in [1] is still missing—the positive recurrence (as opposed to the mere recurrence) of the two particle motion. If this gap is filled then this will be, most likely, sufficient to prove Conjectures 1.5 and 1.6 in [1].

References

[1] Krzysztof Burdzy, Zhen-Qing Chen, and Soumik Pal. Brownian earthworm. *Ann. Probab.*, 41(6):4002–4049, 2013.